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Semi-invariants of mixed representations of quivers

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Abstract

A notion of a mixed representation of quivers can be derived from ordinary quiver representation by considering the dual action of groups on "vertex" vector spaces together with their usual action. A generating system for the algebra of semi-invariants of mixed representations of a quiver is determined. This is done by reducing the problem to the case of bipartite quivers of a special form and using a function DP on three matrices, which is a mixture of the determinant and two pfaffians.

1 Introduction

Quiver representations appeared for the first time in [12]. The importance of this notion is due to the fact that the category of representations of a quiver is equivalent to the category of finite dimensional modules over the path algebra associated with that quiver. Since any finite dimensional basic algebra over an algebraically closed field is a factor-algebra of the path algebra of some quiver (see Chapter 3 of [11]), the set of finite dimensional modules over such an algebra is a full subcategory of the category of representations of the quiver. A representation of a quiver consists of a collection of vector spaces, assigned to its vertices, and linear mappings between the

vector spaces "along" the arrows. A morphism of two representations of the same quiver is a collection of linear mappings between the vector spaces, assigned to the same vertices, which commute with the linear mappings of the representations. The set of quiver representations of a fixed dimension can be endowed naturally with the structure of a vector space and it is called the space of quiver representations of that dimension. Its group of automorphisms is a direct product of the general linear groups, acting on "vertex" spaces. Obviously, the orbits of this action correspond to the classes of isomorphic representations. If we compute generators of the algebra of invariants, then we can distinguish closed orbits, which correspond to semisimple representations.

At first this problem was solved in an important special case of a quiver with one vertex and several loops. Its representation space of dimension n coincides with a space of several $n \times n$ matrices on which $GL(n)$ acts by diagonal conjugations. In the case of a field of characteristic zero, generators for the algebra can easily be found by means of the classical invariant theory (for details see, e.g., [20], page 257). Defining relations between generators were described in [22] and [21]. In the case of positive characteristic, generators for the algebra of invariants were described in [9], and relations between them were established in [26]. Both of these results are formulated in terms of coefficients σ_k in the characteristic polynomial of a matrix. Their proofs rely on the theory of modules with good filtration [8]. Subsequently methods from [9] and [26] were successfully applied to representations of any quiver [10] and [28]. In the case of a field of characteristic zero these results were proved in [17] and later in [6].

Natural generalizations of the above construction are invariants of quivers under the action of classical groups, i.e., of $GL(\mathcal{V})$, $O(\mathcal{V})$, $Sp(\mathcal{V})$, $SL(\mathcal{V})$, and $SO(\mathcal{V})$. In other words, we assume that some classical group acts on the vector space assigned to a vertex, so the product of these groups acts on the space of quiver representations. In particular, the algebra of invariants under the action of a product of the special linear groups is called the algebra of *semi-invariants*. Its generators were calculated in [7] using methods from [9], [10], [26], and [28], and, independently, in [3] and [4] utilizing methods of the representation theory of quivers. In the paper [24] generators for semi-invariants were given in the case of a field of characteristic zero.

For remaining classical groups, the first step was performed in the classical work of Procesi [21]. He calculated generators and relations between them for the orthogonal and symplectic invariants of several matrices over a field of characteristic zero. The study of the case of zero characteristic was continued in [2], where generators for the case of the special orthogonal group were found. The results concerning generators for the orthogonal and symplectic invariants were generalized in [27] to

a field of positive characteristic (assumed to be odd in the case of the orthogonal group). The main reduction of paper [27] (also see [29]) actually shows that the most general concept for dealing with product of arbitrary classical groups is the notion of mixed representations of quivers, introduced in [29].

The main idea of the definition of *mixed representations* is to deal with bilinear forms in addition to linear mappings. It works as follows. Bilinear forms on some vector space $\mathcal{V}$ are in one to one correspondence with linear mappings from $\mathcal{V}$ to its dual vector space $\mathcal{V}^*$. The standard action of $GL(\mathcal{V})$ on $\mathcal{V}$ induces the action on $\mathcal{V}^*$, and, therefore, on $\mathrm{Hom}(\mathcal{V}, \mathcal{V}^*)$. So, together with the action of the general linear groups on the vector spaces assigned to vertices, we consider their dual action.

As an example, consider a quiver $\mathcal{Q}$ that can schematically be depicted as

$$\bullet v \stackrel{a,b}{\Longrightarrow} \bullet w \quad ,$$

where v, w are vertices; a, b are arrows from v to w; a vector space $\mathcal{V}$ is assigned to v, and $\mathcal{V}^*$ is assigned to w. Then the orbits of $GL(\mathcal{V})$ on the space of mixed representations of $\mathcal{Q}$ correspond to pairs of bilinear forms on $\mathcal{V}$. The classification problem for pairs of symmetric and skew-symmetric bilinear forms is a classical topic going back to Weierstrass and Kronecker (see [13], [16], and [15]).

Another motivation for introducing mixed representations are papers [5] and [25], where *orthogonal* and *symplectic* representations of *symmetric* quivers and representations of *signed* quivers, respectively, were introduced. These notions relate to mixed representations of quivers very closely. In fact, such representations are mixed representations for some products of classical linear groups. In [5] it was established that representations of a symmetric quiver are in one-to-one correspondence with a some subset of representations of the associated double quiver. Consequently the symmetric quivers of tame and finite type were classified. These results were generalized to signed quivers in [25].

In [29] generating invariants for mixed representations of quivers were found, as well as invariants for representations of symmetric and signed quivers. In a subsequent paper [30] defining relations for the algebra of invariants of mixed representations of quivers were established. To describe relations, together with σ_k a new function $\sigma_{k,r}$ on three matrices is used. This function describes some relations for the quiver schematically depicted as

$$c \subset \bullet v \stackrel{a,b}{\Longleftrightarrow} \bullet w$$

Here the arrow a goes from v to w, the arrow b goes in the opposite direction, c is a loop on the vertex v, and the vector space assigned to w is dual to the space

assigned to v. The general results were applied to obtain information about defining relations for the orthogonal and symplectic invariants (see Sections 3 and 4 of [30]).

The current paper is dedicated to semi-invariants of mixed representations of quivers. In general we follow the approach of [7]. The main result of the paper is a description of a generating system for semi-invariants of mixed representations of quivers.

Section 2 contains necessary definitions and some auxiliary results.

In Section 3 the function DP on three matrices is introduced and some of its combinatorial properties are studied. This function is a mixture of the determinant and two pfaffians. Note that DP relates to $\sigma_{k,r}$ in the same way as the determinant relates to σ_k, i.e., for $1 \leq k \leq t$ the function $\sigma_{k,r}(X,Y,Z)$ is the coefficient of λ^{t-k} in the polynomial $\mathrm{DP}_{r,r}(X+\lambda E, Y, Z)$ for $(t+2r)\times(t+2r)$ matrices X, Y, Z. (This is a direct consequence of the decomposition formula from [18].)

In Section 4 the general problem is reduced to the case of the so-called zigzag-quivers, which are special cases of bipartite quivers. We prove that semi-invariants of mixed representations of a quiver are spanned by semi-invariants of mixed representations of some zigzag-quiver and describe explicitly this reduction (see Theorem 1).

In Sections 5 and 6 we determine a generating system for semi-invariants of mixed representations of a zigzag-quiver. There is an obvious way to obtain some semi-invariants. First we consider triplets of block matrices, using generic matrices of mixed representations of a quiver as blocks. Partial linearizations of the function DP on these triplets are semi-invariants of mixed representations of a quiver. We prove that all semi-invariants of mixed representations of a quiver belong to the linear span of these semi-invariants (see Theorem 2). In Subsection 5.3 a special case is considered. Theorem 2 is proved in Section 6.

The paper is concluded by example given in Section 7.

Using the main result of this paper together with a reduction to mixed representations of quivers (c.f. [29]) and the decomposition formula from [18], the first author completed description of generators for the invariants under the action of a product of classical groups on the space of (mixed) representations of a quiver in [19]. In particular, generators for the invariants of several matrices under the action of the special orthogonal group were found, where the characteristic is different from 2 when we consider the (special) orthogonal group.

2 Preliminaries

2.1 Notations and remarks

Let $\mathcal{K}$ be an infinite field of arbitrary characteristic. All vector spaces, algebras and modules are over $\mathcal{K}$ unless otherwise stated. Denote by $\mathbb{N}$ the set of all non-negative integers, by $\mathbb{Z}$ the set of all integers, and by $\mathbb{Q}$ the quotient field of $\mathbb{Z}$. We use capital letters like $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc. for sets endowed with some algebraic structure and for quivers (see Section 2.2).

For a vector space $\mathcal{V}$ let $\mathcal{V}^*$, $S(\mathcal{V})$, $S^t(\mathcal{V})$, $\otimes^t\mathcal{V}$, and $\wedge^t\mathcal{V}$, respectively, stand for the dual space, the symmetric algebra, the t-th symmetric power, the t-th tensor power, and the t-th exterior power of $\mathcal{V}$, respectively.

Suppose a reductive algebraic group $\mathcal{G}$ acts on vector spaces $\mathcal{V}$ and $\mathcal{W}$, and let $v\in\mathcal{V}$, $w\in\mathcal{W}$, $g\in\mathcal{G}$. This action induces an action on

- the coordinate algebra $\mathcal{K}[\mathcal{V}]$ of the affine variety $\mathcal{V}$: for $f\in\mathcal{K}[\mathcal{V}]$ we put $(g\cdot f)(v)=f(g^{-1}\cdot v)$;
- the dual space $\mathcal{V}^*$, which we consider as the degree one homogeneous component of the graded algebra $\mathcal{K}[\mathcal{V}]$;
- the space $\mathrm{Hom}_{\mathcal{K}}(\mathcal{V},\mathcal{W})$ of $\mathcal{K}$-linear maps from $\mathcal{V}$ to $\mathcal{W}$: for $H\in\mathrm{Hom}_{\mathcal{K}}(\mathcal{V},\mathcal{W})$ put $(g\cdot H)(v)=g\cdot(H(g^{-1}\cdot v))$;
- the tensor product $\mathcal{V}\otimes\mathcal{W}$: $g\cdot(v\otimes w)=g\cdot v\otimes g\cdot w$.

Given $n\in\mathbb{N}$, consider the vector space $\mathcal{V}(n)=\mathcal{K}^n$ of column vectors of length n with the *standard* basis $v_1,\ldots,v_n$, where v_i is a column vector, whose i-th entry is 1 and the rest of entries are zeros. Consider the dual space $\mathcal{V}(n)^*$ with the dual basis $v_1^*,\ldots,v_n^*$. We identify $\mathcal{V}(n)^*$ with the space of column vectors of length n, so v_i^* is the same column vector as v_i. Denote by $\mathcal{K}^{n\times m}$ the space of $n\times m$ matrices.

The general linear group $GL(n)=\{g\in\mathcal{K}^{n\times n}\mid \det(g)\neq 0\}$ acts on $\mathcal{V}(n)$ by left multiplication. Let $\{1,*\}$ be the group of order two, where the symbol 1 stands for the identity element of the group. Given $\alpha\in\{1,*\}$, $g\in GL(n)$, define

$$\mathcal{V}(n)^\alpha=\left\{\begin{array}{l}\mathcal{V}(n) \text{ for } \alpha=1\\ \mathcal{V}(n)^* \text{ for } \alpha=*\end{array}\right.,\quad v_i^\alpha=\left\{\begin{array}{l}v_i \text{ for } \alpha=1\\ v_i^* \text{ for } \alpha=*\end{array}\right.,\quad g^\alpha=\left\{\begin{array}{cl}g & \text{for } \alpha=1\\ (g^{-1})^T & \text{for } \alpha=*\end{array}\right..$$

Note that

$$g\cdot v_i^\alpha=g^\alpha v_i^\alpha. \tag{1}$$

Let $\alpha,\beta\in\{1,*\}$, $\mathcal{V}=\mathcal{V}(n)^{\alpha}$, $\mathcal{W}=\mathcal{W}(m)^{\beta}$ and let $w_1,\ldots,w_m$ be the standard basis for $\mathcal{W}(m)$. Let groups $\mathcal{G}_1\subset GL(n)$, $\mathcal{G}_2\subset GL(m)$ be closed under transposition and the group $\mathcal{G}=\mathcal{G}_1\times\mathcal{G}_2$ acts on $\mathcal{V}$, $\mathcal{W}$ as follows: for $g=(g_1,g_2)\in\mathcal{G}$, $v\in\mathcal{V}$, $w\in\mathcal{W}$ we have $g\cdot v=g_1\cdot v=g_1^{\alpha}v$, $g\cdot w=g_2\cdot w=g_2^{\beta}w$. Let $\mathcal{K}[\mathrm{Hom}_{\mathcal{K}}(\mathcal{V},\mathcal{W})]=\mathcal{K}[x_{ij}\,|\,1\leq i\leq m,\,1\leq j\leq n]$ be the coordinate algebra of the space $\mathrm{Hom}_{\mathcal{K}}(\mathcal{V},\mathcal{W})$, and $X=(x_{ij})$ be the $m\times n$ matrix. Here x_{ij} maps a matrix $H\in\mathcal{K}^{m\times n}\simeq\mathrm{Hom}_{\mathcal{K}}(\mathcal{V},\mathcal{W})$ to the (i,j)-th entry of H, where the isomorphism is determined by the choice of bases for $\mathcal{V},\mathcal{W}$. Consider $g=(g_1,g_2)\in\mathcal{G}$, $H\in\mathcal{K}^{m\times n}\simeq\mathrm{Hom}_{\mathcal{K}}(\mathcal{V},\mathcal{W})$. The following lemma is proved by straightforward calculations.

Lemma 1 *Using the preceding notation we have:*

1) $g\cdot H=g_2^{\beta}H(g_1^{\alpha})^{-1}$.

2) $g\cdot X=(g_2^{\beta})^{-1}Xg_1^{\alpha}$, *where* $g\cdot X$ *means the matrix whose* (i,j)*-th entry is equal to* $g\cdot x_{ij}$.

3) The homomorphism of spaces $\Phi:\mathcal{K}[\mathrm{Hom}_{\mathcal{K}}(\mathcal{V},\mathcal{W})]\to S(\mathcal{W}(m)^{\beta *}\otimes\mathcal{V}(n)^{\alpha})$, *defined by* $\Phi(x_{ij})=w_i^{\beta *}\otimes v_j^{\alpha}$, *is an isomorphism of* $\mathcal{G}$*-modules.*

Proof. 1) Let $v\in\mathcal{V}$. Then $(g\cdot H)(v)=g\cdot(H(g^{-1}\cdot v))=g\cdot(H(g_1^{-1})^{\alpha}v)=g_2^{\beta}H(g_1^{-1})^{\alpha}v$.

2) For $f\in\mathcal{K}[x_{ij}]$ and $H\in\mathcal{K}^{m\times n}\simeq\mathrm{Hom}_{\mathcal{K}}(\mathcal{V},\mathcal{W})$ we have $(g\cdot f)(H)=f(g^{-1}\cdot H)=f((g_2^{\beta})^{-1}Hg_1^{\alpha})$.

3) We prove that Φ is $\mathcal{G}$-homomorphism. Denote the entries of the matrix g by g_{ij}. By the previous item, $\Phi(g\cdot x_{ij})$ equals

$$\Phi(((g_2^{\beta})^{-1}Xg_1^{\alpha})_{ij})=\Phi(\sum\nolimits_{r,s}((g_2^{\beta})^{-1})_{ir}x_{rs}(g_1^{\alpha})_{sj})=\sum\nolimits_{r,s}((g_2^{\beta})^{-1})_{ir}(g_1^{\alpha})_{sj}w_r^{\beta *}\otimes v_s^{\alpha}.$$

On the other hand, $g\cdot(w_i^{\beta *}\otimes v_j^{\alpha})=g\cdot w_i^{\beta *}\otimes g\cdot v_j^{\alpha}=g_2^{\beta *}w_i^{\beta *}\otimes g_1^{\alpha}v_j^{\alpha}=\sum_{r,s}(g_2^{\beta *})_{ri}(g_1^{\alpha})_{sj}w_r^{\beta *}\otimes v_s^{\alpha}$. The required statement follows from $(g_2^{\beta *})^T=(g_2^{\beta})^{-1}$. $\triangle$

Denote by id_n the identity permutation of $\mathcal{S}_n$. Given permutations $\sigma_1\in\mathcal{S}_{n_1},\ldots,\sigma_k\in\mathcal{S}_{n_k}$, denote the product of their signs $\mathrm{sgn}(\sigma_1)\cdot\ldots\cdot\mathrm{sgn}(\sigma_k)$ by $\mathrm{sgn}(\sigma_1\ldots\sigma_k)$. For positive integers n_1,n_2,n_3 and permutations $\sigma_1\in\mathcal{S}_{n_1},\sigma_2\in\mathcal{S}_{n_2},\sigma_3\in\mathcal{S}_{n_3}$ we regard $\sigma=\sigma_1\times\sigma_2\times\sigma_3$ as an element of $\mathcal{S}_{n_1+n_2+n_3}$ given by

$$\sigma(l)=\left\{\begin{array}{cl}\sigma_1(l), & \text{if}\quad l\leq n_1\\ \sigma_2(l-n_1)+n_1, & \text{if}\quad n_1<l\leq n_1+n_2\\ \sigma_3(l-n_1-n_2)+n_1+n_2, & \text{if}\quad n_1+n_2<l\leq n_1+n_2+n_3.\end{array}\right.$$

For any vector spaces $\mathcal{V}, \mathcal{W}$ and a positive integer n there is a homomorphism of vector spaces

$$\zeta_n : \wedge^n \mathcal{V}^* \otimes \wedge^n \mathcal{W} \to S^n(\mathcal{V}^* \otimes \mathcal{W}),$$

defined by

$$\zeta_n(v_1^* \wedge \ldots \wedge v_n^* \otimes w_1 \wedge \ldots \wedge w_n) = \sum\nolimits_{\rho \in \mathcal{S}_n} \mathrm{sgn}(\rho) \prod\nolimits_{l=1}^{n} v_l^* \otimes w_{\rho(l)},$$

where $v_i^* \in \mathcal{V}^*, w_j \in \mathcal{W}$. Consider an ordered set $\gamma = (\gamma_1, \ldots, \gamma_p) \in \mathbb{N}^p$. We set $\wedge^\gamma \mathcal{V} = \wedge^{\gamma_1} \mathcal{V} \otimes \ldots \otimes \wedge^{\gamma_p} \mathcal{V}$. Define a homomorphism of vector spaces

$$\zeta_\gamma : \wedge^\gamma \mathcal{V}^* \otimes \wedge^\gamma \mathcal{W} \to S^{\gamma_1 + \ldots + \gamma_p}(\mathcal{V}^* \otimes \mathcal{W})$$

by $\zeta_\gamma(c) = \zeta_{\gamma_1}(a_1 \otimes b_1) \ldots \zeta_{\gamma_p}(a_p \otimes b_p)$, where $c = (a_1 \otimes \ldots \otimes a_p) \otimes (b_1 \otimes \ldots \otimes b_p)$, and $a_i \in \wedge^{\gamma_i} \mathcal{V}^*$, $b_i \in \wedge^{\gamma_i} \mathcal{W}$ $(1 \leq i \leq p)$.

The symbol $a \to b$ will indicate a substitution. For $a \in \mathbb{N}$ and $B \subseteq \mathbb{N}$ we write $a + B$ for $\{a + b \,|\, b \in B\}$. Denote the Kronecker symbol by δ_{ij}. Given a group $\mathcal{G}$ and $a, b \in \mathcal{G}$, we set $a^b = b^{-1}ab$. For integers $i < j$ denote the interval $i, i+1, \ldots, j-1, j$ by $[i, j]$. For the cardinality of a set A we write $\#A$.

2.2 The space of mixed representations of a quiver

A *quiver* $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ is a finite oriented graph, where $\mathcal{Q}_0 = \{1, \ldots, l\}$ is the set of vertices and $\mathcal{Q}_1$ is the set of arrows. For an arrow a, denote by a'' its tail, and by a' its head. Consider finite dimensional vector spaces $\mathcal{V}_1 = \mathcal{K}^{n_1}, \ldots, \mathcal{V}_l = \mathcal{K}^{n_l}$ of column vectors and fix their standard bases. The vector $\underline{n} = (n_1, \ldots, n_l)$ is called the *dimension* vector. Consider $\alpha = (\alpha_1, \ldots, \alpha_l)$, where $\alpha_1, \ldots, \alpha_l \in \{1, *\}$ and assign the vector space $\mathcal{V}_u^{\alpha_u}$ to each $u \in \mathcal{Q}_0$.

Further, assume that there is an involution $\phi : \mathcal{Q}_0 \to \mathcal{Q}_0$ satisfying the properties:

a) $n_u = n_{\phi(u)}$;

b) $\phi(u) = u$ implies $\alpha_u = 1$;

c) $\phi(u) \neq u$ implies $\alpha_{\phi(u)} = *\alpha_u$.

In other words, some of the vertices are grouped into non-intersecting pairs, where vector spaces corresponding to a pair of vertices have equal dimensions and one of the vector spaces from such a pair is dual to another whereas all the remaining vertices u have $\alpha_u = 1$ and are stable under ϕ.

Define $GL(\underline{n}) = GL(n_1)\times\ldots\times GL(n_l)$ and

$$\mathcal{R} = \mathcal{R}_{\alpha,\phi}(\mathcal{Q},\underline{n}) = \bigoplus_{a\in\mathcal{Q}_1}\mathcal{K}^{n_{a'}\times n_{a''}} \simeq \bigoplus_{a\in\mathcal{Q}_1}\mathrm{Hom}_{\mathcal{K}}(\mathcal{V}_{a''}^{\alpha_{a''}},\mathcal{V}_{a'}^{\alpha_{a'}}),$$

where the isomorphism is given by the choice of bases for $\mathcal{V}_1,\ldots,\mathcal{V}_l$. For any vertex u vector spaces $\mathcal{V}_u$ and $\mathcal{V}_u^*$ are $GL(n_u)$-modules. Then the group $GL(\underline{n})$ acts on $\mathcal{R}$ by the rule: for $g=(g_1,\ldots,g_l)\in GL(\underline{n})$, $(H_a)_{a\in\mathcal{Q}_1}\in\mathcal{R}$ we have

$$g\cdot(H_a)_{a\in\mathcal{Q}_1} = (g\cdot H_a)_{a\in\mathcal{Q}_1} = (g_{a'}^{\alpha_{a'}}H_a(g_{a''}^{\alpha_{a''}})^{-1})_{a\in\mathcal{Q}_1}$$

(see Lemma 1). For each pair $(u,\phi(u))$ with $u\in\mathcal{Q}_0$ and $\phi(u)\neq u$ replace the factor $GL(n_u)\times GL(n_{\phi(u)})$ of the group $GL(\underline{n})$ by its diagonal subgroup. The resulting subgroup

$$\{g\in GL(\underline{n})\,|\,g_u = g_{\phi(u)} \text{ for all } u\in\mathcal{Q}_0\}$$

is isomorphic to

$$GL_{\alpha,\phi}(\underline{n}) = \prod_{a\in\mathcal{Q}_0,\,\alpha(a)=1} GL(n_a)$$

by $(g_a)_{a\in\mathcal{Q}_0}\mapsto(g_a)_{a\in\mathcal{Q}_0,\,\alpha(a)=1}$. This isomorphism induces the action of $GL_{\alpha,\phi}(\underline{n})$ on $\mathcal{R}$. So, one and the same general linear group acts on both $\mathcal{V}_u$ and $\mathcal{V}_u^*$. The space $\mathcal{R}$ together with the action of $GL_{\alpha,\phi}(\underline{n})$ on it is called a $\underline{n}$-dimensional *space of mixed representations* of $\mathcal{Q}$, and elements of $\mathcal{R}$ are called *mixed representations*.

The coordinate ring of the affine variety $\mathcal{R}$ is the polynomial ring

$$\mathcal{K}[\mathcal{R}] = \mathcal{K}[x_{ij}^b\,|\,b\in\mathcal{Q}_1,\ 1\leq i\leq n_{b'}, 1\leq j\leq n_{b''}].$$

Here x_{ij}^b stands for the coordinate function on $\mathcal{R}$ that takes a representation $H=(H_a)_{a\in\mathcal{Q}_1}$ to the (i,j)-th entry of H_b. The ring $\mathcal{K}[\mathcal{R}]$ is a $GL_{\alpha,\phi}(\underline{n})$-module (Section 2.1). Denote by

$$\mathcal{K}[\mathcal{R}]^{GL_{\alpha,\phi}(\underline{n})} = \{f\in\mathcal{K}[\mathcal{R}]\,|\,g\cdot f = f \text{ for all } g\in GL_{\alpha,\phi}(\underline{n})\}$$

the algebra of *invariants* of mixed representations of the quiver $\mathcal{Q}$.

Let $SL_{\alpha,\phi}(\underline{n}) = GL_{\alpha,\phi}(\underline{n})\cap(SL(n_1)\times\ldots\times SL(n_l))$. Denote by

$$\mathcal{K}[\mathcal{R}]^{SL_{\alpha,\phi}(\underline{n})} = \{f\in\mathcal{K}[\mathcal{R}]\,|\,g\cdot f = f \text{ for all } g\in SL_{\alpha,\phi}(\underline{n})\}$$

the algebra of *semi-invariants* of mixed representations of the quiver.

If $\alpha_u = 1$ for all $u\in\mathcal{Q}_0$, then (semi)-invariants of mixed representations of $\mathcal{Q}$ are (semi)-invariants of representations of $\mathcal{Q}$.

For $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{Z}^m$, where m is the cardinality of $\{u \in \mathcal{Q}_0 \,|\, \alpha_u = 1\}$, denote the space of *relative invariants of weight* ϵ by

$$\mathcal{K}[\mathcal{R}]^{GL_{\alpha,\phi}(\underline{n}),\epsilon} = \{f \in \mathcal{K}[\mathcal{R}] \,|\, g \cdot f = (\prod_{u\in\mathcal{Q}_0,\, \alpha_u=1} \det^{\epsilon_u}(g_u))f \text{ for all } g \in GL_{\alpha,\phi}(\underline{n})\}.$$

2.3 Distributions, partitions and Young subgroups

By a *distribution* $B = (B_1, \ldots, B_d)$ of a set $[1,t]$ we mean an ordered partition of the set $[1,t]$ into pairwise disjoint subsets B_j ($1 \leq j \leq d$), which are called components of the distribution. To every B we associate two functions $l \mapsto B|l|$ and $l \mapsto B\langle l\rangle$ ($1 \leq l \leq t$), defined by the rules:

$$B|l| = i, \text{ if } l \in B_i, \text{ and } B\langle l\rangle = \#\{[1,l] \cap B_i \,|\, l \in B_i\}.$$

The symmetric group $\mathcal{S}_t$ acts on $[1,t]$ and contains the *Young subgroup*

$$\mathcal{S}_B = \{\pi \in \mathcal{S}_t \,|\, \pi(B_i) = B_i \text{ for } 1 \leq i \leq d\}.$$

For $\sigma \in \mathcal{S}_t$ we set $B^\sigma = (B_1^\sigma, \ldots, B_d^\sigma)$, where $B_i^\sigma = \sigma^{-1}B_i$, $1 \leq i \leq d$. The intersection $A \cap B$ of distributions $A = (A_1, \ldots, A_p)$ and $B = (B_1, \ldots, B_q)$ of the same set is the result of pairwise intersections of all components of the given distributions, i.e., $C = (C_1, \ldots, C_d)$, where for any $1 \leq k \leq d$ we have $C_k \neq \emptyset$ and there exist i, j such that $1 \leq i \leq p$, $1 \leq j \leq q$, and $C_k = A_i \cap B_j$; moreover, we assume that for any $1 \leq k_1 < k_2 \leq d$ the minimum of C_{k_1} is less that the minimum of C_{k_2}. If A, B are distributions of the same set, then we write $A \leq B$ provided each component of A is contained in some component of B.

A vector $\underline{t} = (t_1, \ldots, t_d) \in \mathbb{N}^d$ determines the distribution $T = (T_1, \ldots, T_d)$ of the set $[1,t]$, where $t = t_1 + \ldots + t_d$ and $T_i = \{t_1 + \ldots + t_{i-1} + 1, \ldots, t_1 + \ldots + t_i\}$, $1 \leq i \leq d$. We use capital letters to refer to the distribution determined by a vector. Note that for $1 \leq l \leq t$ we have

$$T|l| = i, \text{ if } t_1 + \ldots + t_{i-1} < l \leq t_1 + \ldots + t_i, \text{ and}$$

$$T\langle l\rangle = l - (t_1 + \ldots + t_{i-1}), \text{ where } T|l| = i.$$

It is easy to derive the following lemma (see Section 2.2 of [7]).

Lemma 2 *Given a distribution $B = (B_1, \ldots, B_d)$ of a set $[1,t]$, consider $\underline{t} = (\#B_1, \ldots, \#B_d)$. Let T be the distribution determined by $\underline{t}$. Then for each $\sigma \in \mathcal{S}_t$ we have:*

1. $\mathcal{S}_{B^\sigma} = \sigma^{-1}\mathcal{S}_B\sigma = \mathcal{S}_B^\sigma$, $B^\sigma|l| = B|\sigma(l)|$ *for any* $1 \le l \le t$.

2. There is a permutation $\eta \in \mathcal{S}_B^\sigma$ *with* $B\langle\sigma(l)\rangle = B^\sigma\langle\eta(l)\rangle$ *for any* $1 \le l \le t$.

3. There exists a permutation $\rho \in \mathcal{S}_t$ *with* $B = T^\sigma$ *for any* $\sigma \in \mathcal{S}_T\rho$. *Moreover, there is a unique permutation* $\sigma \in \mathcal{S}_T\rho$ *that satisfies the condition* $T^\sigma\langle l\rangle = T\langle\sigma(l)\rangle$ *for any* $1 \le l \le t$.

A vector $\gamma = (\gamma_1, \ldots, \gamma_p) \in \mathbb{N}^p$ satisfying $\gamma_1 \ge \ldots \ge \gamma_p$ and $\gamma_1 + \ldots + \gamma_p = t$ is called a *partition* of $t \in \mathbb{N}$ and is denoted by $\gamma \vdash t$. A *multipartition* $\gamma \vdash \underline{t}$ is a d-tuple of partitions $\gamma = (\gamma(1), \ldots, \gamma(d))$, where $\gamma(i) \vdash t_i$, $\underline{t} = (t_1, \ldots, t_d)$. We identify a multipartition γ with a vector $(\gamma_1, \ldots, \gamma_p) \in \mathbb{N}^p$, where $\gamma(i) = (\gamma_{p_1+\ldots+p_{i-1}+1}, \ldots, \gamma_{p_1+\ldots+p_i})$, $p = p_1 + \ldots + p_d$, and say that $\underline{p} = (p_1, \ldots, p_d)$ is the *height* of γ. Any multipartition $\gamma \in \mathbb{N}^p$ determines the distribution $\Gamma = (\Gamma_1, \ldots, \Gamma_p)$ and the Young subgroup $\mathcal{S}_\Gamma$, as above.

For a subgroup $\mathcal{A}$ of $\mathcal{G}$ denote by $\mathcal{G}/\mathcal{A}$, $\mathcal{A}\backslash\mathcal{G}$ and $\mathcal{A}\backslash\mathcal{G}/B$, respectively, some system of representatives for the left, right and double cosets respectively. Note that if σ ranges over $\mathcal{G}/\mathcal{A}$, then σ^{-1} ranges over $\mathcal{A}\backslash\mathcal{G}$ and vice versa.

Lemma 3 *Let* $\mathcal{G}$ *be a group, and let* $\mathcal{L}, \mathcal{A}, \mathcal{B}$ *be its subgroups. Then*

a) for any system of representatives $\mathcal{L}/\mathcal{L}\cap\mathcal{B}$ *there exist systems of representatives* $\mathcal{L}\cap\mathcal{A}\backslash\mathcal{L}/\mathcal{L}\cap\mathcal{B}$ *and* $\mathcal{A}\cap\mathcal{L}/\mathcal{A}\cap\mathcal{L}\cap\mathcal{B}^\pi$, *where* $\pi^{-1} \in \mathcal{L}\cap\mathcal{A}\backslash\mathcal{L}/\mathcal{L}\cap\mathcal{B}$, *such that*

$$\mathcal{L}/\mathcal{L}\cap\mathcal{B} = \{\nu\pi^{-1} \,|\, \pi^{-1} \in \mathcal{L}\cap\mathcal{A}\backslash\mathcal{L}/\mathcal{L}\cap\mathcal{B},\, \nu \in \mathcal{A}\cap\mathcal{L}/\mathcal{A}\cap\mathcal{L}\cap\mathcal{B}^\pi\}; \qquad (2)$$

b) for any systems of representatives $\mathcal{L}\cap\mathcal{A}\backslash\mathcal{L}/\mathcal{L}\cap\mathcal{B}$ *and* $\mathcal{A}\cap\mathcal{L}/\mathcal{A}\cap\mathcal{L}\cap\mathcal{B}^\pi$, *where* $\pi^{-1} \in \mathcal{L}\cap\mathcal{A}\backslash\mathcal{L}/\mathcal{L}\cap\mathcal{B}$, *there exist a system of representatives* $\mathcal{L}/\mathcal{L}\cap\mathcal{B}$ *such that (2) holds.*

Proof. **a)** Consider a set $\mathcal{L}/\mathcal{L}\cap\mathcal{B} = \{g_i \,|\, i \in I\}$. Since $\mathcal{L}$ is a union of double cosets, for each $i \in I$ there is a $\pi_i \in \mathcal{L}$ such that $g_i \in (\mathcal{L}\cap\mathcal{A})\pi_i^{-1}(\mathcal{L}\cap\mathcal{B})$. Multiplying g_i by an appropriate element from $\mathcal{L}\cap\mathcal{B}$ we can assume that $g_i = \nu\pi_i^{-1}$ for some $\nu \in \mathcal{L}\cap\mathcal{A}$ for all i. If two elements $\nu\pi^{-1}$ and $\nu'\pi'^{-1}$ from $\{\nu\pi^{-1} \,|\, \pi^{-1} \in \mathcal{L}\cap\mathcal{A}\backslash\mathcal{L}/\mathcal{L}\cap\mathcal{B}, \nu \in \mathcal{A}\cap\mathcal{L}\}$ are equal modulo $\mathcal{L}\cap\mathcal{B}$, then $(\mathcal{L}\cap\mathcal{A})\pi^{-1}(\mathcal{L}\cap\mathcal{B}) = (\mathcal{L}\cap\mathcal{A})\pi'^{-1}(\mathcal{L}\cap\mathcal{B})$, and thus $\pi = \pi'$. Therefore $\nu\pi^{-1} \equiv \nu'\pi'^{-1} \bmod(\mathcal{L}\cap\mathcal{B})$ holds if and only if $\nu^{-1}\nu' \in \mathcal{L}\cap\mathcal{A}\cap\mathcal{B}^\pi$.

b) Consider sets $\mathcal{L}\cap\mathcal{A}\backslash\mathcal{L}/\mathcal{L}\cap\mathcal{B} = \{\pi_i^{-1} \,|\, i \in I\}$ and $\mathcal{A}\cap\mathcal{L}/\mathcal{A}\cap\mathcal{L}\cap\mathcal{B}^{\pi_i} = \{\nu_{ij} \,|\, j \in J_i\}$ for any $i \in I$. We should prove that $C = \{\nu_{ij}\pi_i^{-1} \,|\, i \in I,\, j \in J_i\}$ is a system of representatives $\mathcal{L}/\mathcal{L}\cap\mathcal{B}$.

For any $g \in \mathcal{L}$ there is an $i \in I$ such that $g = a\pi_i^{-1}b$ for some $a \in \mathcal{L} \cap \mathcal{A}$, $b \in \mathcal{L} \cap \mathcal{B}$. Hence $a = \nu_{ij}a'$ for some $j \in J_i$, $a' \in \mathcal{A} \cap \mathcal{L} \cap \mathcal{B}^{\pi_i}$. Therefore $a' = \pi_i^{-1}b'\pi_i$ for some $b' \in \mathcal{B}$. Finally, $g = \nu_{ij}\pi_i^{-1}b'b$ is contained in C modulo $\mathcal{L} \cap \mathcal{B}$. Repeating the reasoning from part a) we obtain that all elements of C are different modulo $\mathcal{L} \cap \mathcal{B}$. $\triangle$

3 Definition and properties of DP

In this section we assume that all matrices have entries in a commutative unitary ring $\mathcal{F}$ without divisors of zero. Moreover, we assume that the characteristic of $\mathcal{F}$ is zero. The case of positive characteristic is discussed in Remark 1 at the end of this section.

Before defining the function DP recall the notion of the pfaffian and the determinant. The pfaffian of a $2r \times 2r$ skew-symmetric matrix $C = (c_{ij})$ is given by

$$\mathrm{pf}(C) = \sum_{\{\sigma \in \mathcal{S}_{2r} \,|\, \sigma(2k-1) < \sigma(2k) \text{ for all } 1 \le k \le r\}} \mathrm{sgn}(\sigma) \prod_{k=1}^{r} c_{\sigma(2k-1),\sigma(2k)}.$$

By the *generalized pfaffian* of an arbitrary $2r \times 2r$ matrix $Y = (y_{ij})$ we will mean

$$\mathrm{P}(Y) = \sum_{\sigma \in \mathcal{S}_{2r} / \operatorname{diag}(\mathcal{S}_r \times \mathcal{S}_r)} \mathrm{sgn}(\sigma) \prod_{k=1}^{r} y_{\sigma(k),\sigma(k+r)},$$

where $\operatorname{diag}(\mathcal{S}_r \times \mathcal{S}_r) = \{\nu \times \nu \,|\, \nu \in \mathcal{S}_r\}$ is the diagonal subgroup of $\mathcal{S}_r \times \mathcal{S}_r$. By abuse of notation we will also refer to $\mathrm{P}(Y)$ as the pfaffian. Note that

$$\mathrm{pf}(Y - Y^T) = (-1)^{r(r-1)/2} r!\, \mathrm{P}(Y).$$

The determinant of a $t \times t$ matrix $X = (x_{ij})$ is equal to

$$\det(X) = \sum_{\sigma \times \tau \in \mathcal{S}_t \times \mathcal{S}_t / \operatorname{diag}(\mathcal{S}_t \times \mathcal{S}_t)} \mathrm{sgn}(\sigma\tau) \prod_{k=1}^{t} x_{\sigma(k),\tau(k)}.$$

Definition 1. Let $t, r, s \in \mathbb{N}$. Given

- a $(t + 2r) \times (t + 2s)$ matrix $X = (x_{ij})$,
- a $(t + 2r) \times (t + 2r)$ matrix $Y = (y_{ij})$,

- a $(t+2s)\times(t+2s)$ matrix $Z=(z_{ij})$,

we define a function $\mathrm{DP}_{r,s}(X,Y,Z)$, which is a mixture of the determinant and two pfaffians:

$$\sum_{\sigma\times\tau\in\mathcal{S}_{t+2r}\times\mathcal{S}_{t+2s}/\mathcal{P}} \mathrm{sgn}(\sigma\tau)\prod_{i=1}^{t} x_{\sigma(i),\tau(i)} \prod_{j=1}^{r} y_{\sigma(t+j),\sigma(t+r+j)} \prod_{k=1}^{s} z_{\tau(t+k),\tau(t+s+k)}, \qquad (3)$$

where $\mathcal{P}=\{\nu_1\times\nu_2\times\nu_2\times\nu_1\times\nu_3\times\nu_3\,|\,\nu_1\in\mathcal{S}_t,\nu_2\in\mathcal{S}_r,\nu_3\in\mathcal{S}_s\}$. It is not difficult to see that $\mathrm{DP}_{r,s}(X,Y,Z)$ can also be written in the form

$$\frac{1}{t!r!s!}\sum_{\sigma\in\mathcal{S}_{t+2r}}\sum_{\tau\in\mathcal{S}_{t+2s}} \mathrm{sgn}(\sigma\tau)\prod_{i=1}^{t} x_{\sigma(i),\tau(i)} \prod_{j=1}^{r} y_{\sigma(t+j),\sigma(t+r+j)} \prod_{k=1}^{s} z_{\tau(t+k),\tau(t+s+k)}.$$

In the case $t=r=s=0$ we define $\mathrm{DP}_{r,s}(X,Y,Z)=0$.

Lemma 4 *The function* DP *satisfies the following properties:*

1.*a*) $\mathrm{DP}_{r,s}(X^T,Z,Y)=\mathrm{DP}_{r,s}(X,Y,Z)$.

b) $\mathrm{DP}_{r,s}(X,Y^T,Z)=(-1)^r\mathrm{DP}_{r,s}(X,Y,Z)$,
$\mathrm{DP}_{r,s}(X,Y,Z^T)=(-1)^s\mathrm{DP}_{r,s}(X,Y,Z)$.

c) *if* $r=s=0$, *then* $\mathrm{DP}_{0,0}(X,Y,Z)=\det(X)$; *and if* $t=0$, *then* $\mathrm{DP}_{r,s}(X,Y,Z)=\mathrm{P}(Y)\mathrm{P}(Z)$.

2. *Let* g *be a* $(t+2r)\times(t+2r)$ *matrix and* h *be a* $(t+2s)\times(t+2s)$ *matrix. Then*

a) $\mathrm{DP}_{r,s}(gX,gYg^T,Z)=\det(g)\mathrm{DP}_{r,s}(X,Y,Z)$.

b) $\mathrm{DP}_{r,s}(Xh,Y,h^TZh)=\det(h)\mathrm{DP}_{r,s}(X,Y,Z)$.

Proof. Part 1 follows easily from the definition of DP.

Given mappings $a:[1,t]\to[1,t+2r]$ and $b,c:[1,r]\to[1,t+2r]$, set

$$M_{a,b,c}=\sum_{\sigma\in\mathcal{S}_{t+2r}}\sum_{\tau\in\mathcal{S}_{t+2s}}\mathrm{sgn}(\sigma\tau)$$

$$\prod_{i=1}^{t} x_{a(i),\tau(i)} \prod_{j=1}^{r} y_{b(j),c(j)} \prod_{k=1}^{s} z_{\tau(t+k),\tau(t+s+k)} \prod_{i=1}^{t} g_{\sigma(i),a(i)} \prod_{j=1}^{r} g_{\sigma(t+j),b(j)} g_{\sigma(t+r+j),c(j)}.$$

Then

$$\mathrm{DP}_{r,s}(gX,gYg^T,Z)=\frac{1}{t!r!s!}\sum_{a,b,c} M_{a,b,c},$$

where the sum is over all mappings $a:[1,t]\to[1,t+2r]$ and $b,c:[1,r]\to[1,t+2r]$.

We denote by (i,j) the transposition switching i and j and make the following observations:

(i) If there are u,v with $1\leq u,v\leq t$ such that $u\neq v$ and $a(u)=a(v)$, then $M_{a,b,c}=0$ because $\mathrm{sgn}(\sigma)\neq\mathrm{sgn}(\sigma\cdot(u,v))$.

(ii) If there are u,v with $1\leq u,v\leq r$ such that $u\neq v$ and $b(u)=b(v)$, then $M_{a,b,c}=0$ because $\mathrm{sgn}(\sigma)\neq\mathrm{sgn}(\sigma\cdot(t+u,t+v))$.

(iii) If there are u,v with $1\leq u,v\leq r$ such that $u\neq v$ and $c(u)=c(v)$, then $M_{a,b,c}=0$ because $\mathrm{sgn}(\sigma)\neq\mathrm{sgn}(\sigma\cdot(t+r+u,t+r+v))$.

(iv) If there are u,v with $1\leq u,v\leq r$ such that $b(u)=c(v)$, then $M_{a,b,c}=0$ because $\mathrm{sgn}(\sigma)\neq\mathrm{sgn}(\sigma\cdot(t+u,t+r+v))$.

(v) If there are u,v with $1\leq u\leq t$ and $1\leq v\leq r$ such that $a(u)=b(v)$, then $M_{a,b,c}=0$ because $\mathrm{sgn}(\sigma)\neq\mathrm{sgn}(\sigma\cdot(u,t+v))$.

(vi) If there are u,v with $1\leq u\leq t$ and $1\leq v\leq r$ such that $a(u)=c(v)$, then $M_{a,b,c}=0$ because $\mathrm{sgn}(\sigma)\neq\mathrm{sgn}(\sigma\cdot(u,t+r+v))$.

Assume $M_{a,b,c}\neq 0$. By (i)–(iii) the mappings a,b,c are injective and by (iv)–(vi) their images are disjoint. Thus there is a $\pi\in\mathcal{S}_{t+2r}$ such that for every $1\leq i\leq t$, $1\leq j\leq r$ we have $\pi(i)=a(i)$, $\pi(t+j)=b(j)$, $\pi(t+r+j)=c(j)$. Therefore

$$\mathrm{DP}_{r,s}(gX,gYg^T,Z)=\frac{1}{t!r!s!}\sum_{\pi\in\mathcal{S}_{t+2r}}\sum_{\sigma\in\mathcal{S}_{t+2r}}\sum_{\tau\in\mathcal{S}_{t+2s}}\mathrm{sgn}(\sigma\tau)$$

$$\prod_{i=1}^{t}x_{\pi(i),\tau(i)}\prod_{j=1}^{r}y_{\pi(t+j),\pi(t+r+j)}\prod_{k=1}^{s}z_{\tau(t+k),\tau(t+s+k)}\prod_{i=1}^{t+2r}g_{\sigma\pi^{-1}(i),i}.$$

Substituting $\sigma\pi$ for σ, we prove 2a). Part 2b) is analogous. $\triangle$

Let $\gamma\vdash t$, $\delta\vdash r$, $\lambda\vdash s$, where $\gamma=(\gamma_1,\ldots,\gamma_u)$, $\delta=(\delta_1,\ldots,\delta_v)$, $\lambda=(\lambda_1,\ldots,\lambda_w)$, and let

- $X_k=(x_{ij}^k)$ be a $(t+2r)\times(t+2s)$ matrix, where $1\leq k\leq u$,
- $Y_k=(y_{ij}^k)$ be a $(t+2r)\times(t+2r)$ matrix, where $1\leq k\leq v$,

- $Z_k = (z_{ij}^k)$ be a $(t+2s)\times(t+2s)$ matrix, where $1\leq k\leq w$.

Consider the polynomial $\mathrm{DP}_{r,s}(x_1X_1+\ldots+x_uX_u, y_1Y_1+\ldots+y_vY_v, z_1Z_1+\ldots+z_wZ_w)$ in variables $x_1,\ldots,x_u$, $y_1,\ldots,y_v$, $z_1,\ldots,z_w$. Denote by

$$\mathrm{DP}_{\gamma,\delta,\lambda}(X_1,\ldots,X_u,Y_1,\ldots,Y_v,Z_1,\ldots,Z_w)$$

the coefficient of $x_1^{\gamma_1}\ldots x_u^{\gamma_u}y_1^{\delta_1}\ldots y_v^{\delta_v}z_1^{\lambda_1}\ldots z_w^{\lambda_w}$ in this polynomial. The function $\mathrm{DP}_{\gamma,\delta,\lambda}$ plays a key role in the rest of the paper.

Lemma 5 *The polynomial* $\mathrm{DP}_{\gamma,\delta,\lambda}(X_1,\ldots,X_u,Y_1,\ldots,Y_v,Z_1,\ldots,Z_w)$ *is equal to*

$$\sum_{\sigma\times\tau\in\mathcal{S}_{t+2r}\times\mathcal{S}_{t+2s}/\mathcal{L}}\mathrm{sgn}(\sigma\tau)\prod_{i=1}^{t}x_{\sigma(i),\tau(i)}^{\Gamma|i|}\prod_{j=1}^{r}y_{\sigma(t+j),\sigma(t+r+j)}^{\Delta|j|}\prod_{k=1}^{s}z_{\tau(t+k),\tau(t+s+k)}^{\Lambda|k|},$$

where Γ,Λ,Δ *are the distributions determined by* γ,δ,λ*, respectively, and*

$$\mathcal{L}=\{\nu_1\times\nu_2\times\nu_2\times\nu_1\times\nu_3\times\nu_3\,|\,\nu_1\in\mathcal{S}_\Gamma,\nu_2\in\mathcal{S}_\Delta,\nu_3\in\mathcal{S}_\Lambda\}.$$

Proof. We set

$$\mathcal{A}=\{a:[1,t]\to[1,u],\ \text{where}\ \#a^{-1}(i)=\gamma_i\ \text{for any}\ 1\leq i\leq u\},$$

$$\mathcal{B}=\{b:[1,r]\to[1,v],\ \text{where}\ \#b^{-1}(j)=\delta_j\ \text{for any}\ 1\leq j\leq v\},$$

$$\mathcal{C}=\{c:[1,s]\to[1,w],\ \text{where}\ \#c^{-1}(k)=\lambda_k\ \text{for any}\ 1\leq k\leq w\}.$$

It is easy to see that

$$\mathrm{DP}_{\gamma,\delta,\lambda}(X_1,\ldots,X_u,Y_1,\ldots,Y_v,Z_1,\ldots,Z_w)$$

$$=1/t!r!s!\sum_{\sigma\in\mathcal{S}_{t+2r}}\sum_{\tau\in\mathcal{S}_{t+2s}}\mathrm{sgn}(\sigma\tau)\sum_{a,b,c}\prod_{i=1}^{t}x_{\sigma(i),\tau(i)}^{a(i)}\prod_{j=1}^{r}y_{\sigma(t+j),\sigma(t+r+j)}^{b(j)}\prod_{k=1}^{s}z_{\tau(t+k),\tau(t+s+k)}^{c(k)},$$

where a,b,c range over $\mathcal{A},\mathcal{B},\mathcal{C}$, respectively. Part 3 of Lemma 2 gives a bijection between the set $\mathcal{S}_t/\mathcal{S}_\Gamma$ and $\mathcal{A}$ that sends each $\pi_1\in\mathcal{S}_t/\mathcal{S}_\Gamma$ to the mapping $i\mapsto\Gamma|\pi_1^{-1}(i)|$. Similarly we obtain bijections between $\mathcal{S}_r/\mathcal{S}_\Delta$ and $\mathcal{S}_s/\mathcal{S}_\Lambda$, respectively, and $\mathcal{B}$ and $\mathcal{C}$, respectively. Thus

$$\mathrm{DP}_{\gamma,\delta,\lambda}(X_1,\ldots,X_u,Y_1,\ldots,Y_v,Z_1,\ldots,Z_w)=\frac{1}{t!r!s!}\sum_{\sigma\in\mathcal{S}_{t+2r}}\sum_{\tau\in\mathcal{S}_{t+2s}}\mathrm{sgn}(\sigma\tau)$$

$$\sum_{\pi_1\in\mathcal{S}_t/\mathcal{S}_\Gamma}\sum_{\pi_2\in\mathcal{S}_r/\mathcal{S}_\Delta}\sum_{\pi_3\in\mathcal{S}_s/\mathcal{S}_\Lambda}\prod_{i=1}^{t}x_{\sigma(i),\tau(i)}^{\Gamma|\pi_1^{-1}(i)|}\prod_{j=1}^{r}y_{\sigma(t+j),\sigma(t+r+j)}^{\Delta|\pi_2^{-1}(j)|}\prod_{k=1}^{s}z_{\tau(t+k),\tau(t+s+k)}^{\Lambda|\pi_3^{-1}(k)|}.$$

After substitution $i\to\pi_1(i), j\to\pi_2(j), k\to\pi_3(k)$ we obtain the expression

$$\frac{1}{t!r!s!\,\#\mathcal{S}_\Gamma\,\#\mathcal{S}_\Delta\,\#\mathcal{S}_\Lambda}\sum_{\pi_1\in\mathcal{S}_t}\sum_{\pi_2\in\mathcal{S}_r}\sum_{\pi_3\in\mathcal{S}_s}\sum_{\sigma\in\mathcal{S}_{t+2r}}\sum_{\tau\in\mathcal{S}_{t+2s}}\mathrm{sgn}(\sigma\tau)$$

$$\prod_{i=1}^{t}x_{\sigma(\pi_1(i)),\tau(\pi_1(i))}^{\Gamma|i|}\prod_{j=1}^{r}y_{\sigma(t+\pi_2(j)),\sigma(t+r+\pi_2(j))}^{\Delta|j|}\prod_{k=1}^{s}z_{\tau(t+\pi_3(k)),\tau(t+s+\pi_3(k))}^{\Lambda|k|}.$$

Another substitution $\sigma\cdot(\pi_1\times\pi_2\times\pi_2)\to\sigma$, $\tau\cdot(\pi_1\times\pi_3\times\pi_3)\to\tau$ gives the final form

$$\frac{1}{\#\mathcal{S}_\Gamma\,\#\mathcal{S}_\Delta\,\#\mathcal{S}_\Lambda}\sum_{\sigma\in\mathcal{S}_{t+2r}}\sum_{\tau\in\mathcal{S}_{t+2s}}\mathrm{sgn}(\sigma\tau)\prod_{i=1}^{t}x_{\sigma(i),\tau(i)}^{\Gamma|i|}\prod_{j=1}^{r}y_{\sigma(t+j),\sigma(t+r+j)}^{\Delta|j|}\prod_{k=1}^{s}z_{\tau(t+k),\tau(t+s+k)}^{\Lambda|k|}.$$

$\triangle$

Denote by $\mathcal{F}[x_1,\ldots,x_n]$ and $\mathbb{Q}[x_1,\ldots,x_n]$, respectively, the ring of commutative polynomials over $\mathcal{F}$ and $\mathbb{Q}$, respectively. The following lemma is trivial.

Lemma 6 *Let $p>0$ be the characteristic of $\mathcal{F}$. Define the ring homomorphism $\pi:\mathbb{Z}\to\mathcal{F}$ by $\pi(1)=1_\mathcal{F}$, where $1_\mathcal{F}$ stands for the identity of $\mathcal{F}$. Consider $a=\sum\alpha_i a_i\in\mathcal{F}[x_1,\ldots,x_n]$, where $\alpha_i\in\pi(\mathcal{F})$, and a_i is a monomial in $x_1,\ldots,x_n$. Take $\beta_i\in\{0,1,\ldots,p-1\}\subset\mathbb{Z}$ such that $\pi(\beta_i)=\alpha_i$ and set $b=\sum_i\beta_i a_i\in\mathbb{Q}[x_1,\ldots,x_n]$.*
Then $b=0$ implies $a=0$.

We will use the following remark to treat the case of positive characteristic of $\mathcal{F}$.

Remark 1 *When $\mathcal{F}$ has a positive characteristic, define* DP *by Formula (3). Then Lemmas 4, 5 remain valid.*

Proof. Analogues of Lemmas 4 and 5 assert equality of certain polynomials over $\pi(\mathcal{F})$ with π defined in Lemma 6. Since we have shown that the corresponding equalities are valid over $\mathbb{Z}$, the proof of Remark 1 follows from Lemma 6. $\triangle$

4 Reduction to a zigzag-quiver

Throughout this section we use notations from Section 2.2.

Definition 2. A quiver $\mathcal{Q}$ is called *bipartite*, if every vertex is a source (i.e. there is no arrow ending at this vertex), or a sink (i.e. there is no arrow starting at this vertex). A *zigzag-quiver* is a bipartite quiver $\mathcal{Q}$ that satisfies the following conditions:

a) for each vertex $u \in \mathcal{Q}_0$ we have $\phi(u) \neq u$; moreover, if u is a source, then $\phi(u)$ is a sink and vice-versa;

b) there is no arrow $a \in \mathcal{Q}_1$ for which $\alpha_{a'} = \alpha_{a''} = *$.

Given a zigzag-quiver $\mathcal{Q}$, we can depict it schematically as

$$\begin{array}{ccc} *\bullet & & \bullet \\ \vdots & \longrightarrow & \vdots \\ *\bullet & & \bullet \\ & \nearrow & \\ \bullet & & \bullet * \\ \vdots & \longrightarrow & \vdots \\ \bullet & & \bullet * \end{array}$$

where the involution ϕ permutes vertices horizontally, and $\alpha_u = *$ if and only if the vertex u is marked on the picture by an asterisk. (For more details see the beginning of Section 5).

Given a quiver $\mathcal{Q}$ and $\underline{n}, \alpha, \phi$, satisfying requirements a), b), c) from Section 2.2, we consider the zigzag-quiver $\mathcal{Q}^{(2)}$ together with $\underline{n}^{(2)}, \alpha^{(2)}, \phi^{(2)}$, according to the following construction, which is a variation of "doubling construction" from [23].

Construction 1. If $\phi(u) = u$, that is a vertex u does not belong to any pair, then we add a new vertex $\overline{u}$, which will form a pair with u, and change $\underline{n}$, ϕ, α, $SL_{\alpha,\phi}(\underline{n})$ appropriately. The resulting space of mixed representations of a new quiver coincides with the space of mixed representations of the original quiver, because there is no arrow whose head or tail is $\overline{u}$. So without loss of generality we can assume that each vertex $u \in \mathcal{Q}_0$ belongs to some pair.

Assume $\phi(u) = v \neq u$, where $u, v \in \mathcal{Q}_0$ and $\alpha_u = 1$, $\alpha_v = *$. Add two new vertices $\overline{u}$, $\overline{v}$ to the quiver and change every arrow a with $a'' = u$ to a new arrow $\overline{a}$ with $\overline{a}'' = \overline{u}$ and $\overline{a}' = a'$. Also change every arrow a with $a'' = v$ to a new arrow $\overline{a}$

with $\overline{a}'' = \overline{v}$ and $\overline{a}' = a'$. We will refer to such arrows as arrows of *type* 1. Define $\alpha_{\overline{u}} = 1$, $\alpha_{\overline{v}} = *$ and $n_{\overline{u}} = n_{\overline{v}} = n_u$. Moreover, add a new arrow b from $\overline{u}$ to u. We will refer to b as an arrow of *type* 2. Redefine ϕ so that $\phi(u) = \overline{v}$, $\phi(v) = \overline{u}$, and change $SL_{\alpha,\phi}(\underline{n})$ appropriately.

After performing this procedure for all pairs, we get the quiver $\mathcal{Q}^{(1)} = (\mathcal{Q}^{(1)}_0, \mathcal{Q}^{(1)}_1)$. If an arrow $a \in \mathcal{Q}^{(1)}_1$ satisfies the condition $\alpha_{a'} = \alpha_{a''} = *$, change it to a new arrow c such that $c' = \phi^{(1)}(a'')$, $c'' = \phi^{(1)}(a')$. We will refer to c as an arrow of *type* 3. After all of these changes we obtain a zigzag-quiver $\mathcal{Q}^{(2)} = (\mathcal{Q}^{(2)}_0, \mathcal{Q}^{(2)}_1)$ and its space of mixed representations $\mathcal{R}_{\alpha^{(2)},\phi^{(2)}}(\mathcal{Q}^{(2)}, \underline{n}^{(2)})$. Denote the coordinate functions on $\mathcal{R}_{\alpha^{(2)},\phi^{(2)}}(\mathcal{Q}^{(2)}, \underline{n}^{(2)})$ by y^a_{ij} $(a \in \mathcal{Q}^{(2)}_1)$.

For short we set $SL^{(1)} = SL_{\alpha^{(1)},\phi^{(1)}}(\underline{n}^{(1)})$ and $SL^{(2)} = SL_{\alpha^{(2)},\phi^{(2)}}(\underline{n}^{(2)})$.

Example. Let $\mathcal{Q}$ be the quiver with vertices v, w, and arrows $a_1, \ldots, a_4$, where a_1 goes from v to w, a_2 goes in the opposite direction, a_3 and a_4 are loops in vertices v and w, respectively. Suppose the involution ϕ interchanges vertices v and w, $\alpha_v = 1$ and $\alpha_w = *$. The quiver $\mathcal{Q}$ is depicted schematically as

$$a_3 \subset \bullet v \overset{a_1,a_2}{\Longleftrightarrow} \bullet w \supset a_4 \quad .$$

Then $\mathcal{Q}^{(2)}$ is

$$\begin{array}{ccc}
\overline{w}\bullet & \overset{a_2}{\longrightarrow} & \bullet v \\
 & \overset{a_3,a_4,b}{\nearrow} & \\
\overline{v}\bullet & \overset{a_1}{\longrightarrow} & \bullet w
\end{array} \quad ,$$

where the arrows of $\mathcal{Q}^{(2)}$ are denoted by the same letters as the corresponding arrows of $\mathcal{Q}$, b is a new arrow; the involution $\phi^{(2)}$ permutes vertices horizontally; and for a vertex i of the new quiver $\alpha^{(2)}_i = *$ if and only if $i = w, \overline{w}$. Note that a_1, a_2, a_3 are arrows of type 1, b is of type 2, and a_4 is of type 3.

The following theorem shows that for our purpose it is enough to find out generators for zigzag-quivers.

Theorem 1 *Consider the homomorphism of $\mathcal{K}$-algebras*

$$\Phi : \mathcal{K}[\mathcal{R}_{\alpha^{(2)},\phi^{(2)}}(\mathcal{Q}^{(2)}, \underline{n}^{(2)})] \to \mathcal{K}[\mathcal{R}_{\alpha,\phi}(\mathcal{Q}, \underline{n})],$$

defined by

$$\Phi(y^a_{ij}) = \left\{ \begin{array}{ll} x^a_{ij}, & \text{if the type of } a \text{ is } 1 \\ \delta_{ij}, & \text{if the type of } a \text{ is } 2 \\ x^a_{ji}, & \text{if the type of } a \text{ is } 3. \end{array} \right.$$

Then its restriction to the algebra of semi-invariants

$$\mathcal{K}[\mathcal{R}_{\alpha^{(2)},\phi^{(2)}}(\mathcal{Q}^{(2)},\underline{n}^{(2)})]^{SL^{(2)}}\to\mathcal{K}[\mathcal{R}_{\alpha,\phi}(\mathcal{Q},\underline{n})]^{SL_{\alpha,\phi}(\underline{n})},$$

is a surjective mapping.

Proof. By Lemma 1 it is enough to prove the claim for the quiver $\mathcal{Q}^{(1)}$ instead of $\mathcal{Q}^{(2)}$. Moreover, we can assume that the procedure of vertex doubling was applied to a single pair of vertices $\{u,v\}$. Let $b\in\mathcal{Q}_1^{(1)}$ be the arrow added to $\mathcal{Q}$ (see the construction of $\mathcal{Q}^{(1)}$).

We have $SL^{(1)}=\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{H}$, $SL_{\alpha,\phi}(\underline{n})=\mathcal{G}\times\mathcal{H}$ for some groups $\mathcal{G},\mathcal{G}_1,\mathcal{G}_2$ that are copies of $SL(n_u)$, i.e., they are equal to $SL(n_u)$, and some group $\mathcal{H}$. The group $\mathcal{G}$ acts on $\mathcal{V}_u$ and $\mathcal{V}_v$, the group $\mathcal{G}_1$ acts on $\mathcal{V}_u^{(1)}$ and $\mathcal{V}_{\overline{v}}^{(1)}$, and the group $\mathcal{G}_2$ acts on $\mathcal{V}_{\overline{u}}^{(1)}$ and $\mathcal{V}_v^{(1)}$. The group $\mathcal{G}$ is embedded into $\mathcal{G}_1\times\mathcal{G}_2$ as the diagonal subgroup

$$\mathcal{D}=\mathrm{diag}(\mathcal{G}_1\times\mathcal{G}_2)=\{(g,g)\,|\,g\in SL(n_u)\}.$$

Let $\mathcal{Q}^{(1)}\backslash\{b\}$ be a quiver obtained by removing the arrow b from $\mathcal{Q}^{(1)}$ and put $\mathcal{R}=\mathcal{R}_{\alpha^{(1)},\phi^{(1)}}(\mathcal{Q}^{(1)}\backslash\{b\},\underline{n}^{(1)})$. The Frobenius reciprocity ([14], Lemma 8.1) implies that

$$\Psi:\left(\mathcal{K}[\mathcal{R}]\otimes\mathcal{K}\left[SL^{(1)}/\mathcal{D}\times\mathcal{H}\right]\right)^{SL^{(1)}}\to\mathcal{K}[\mathcal{R}]^{\mathcal{D}\times\mathcal{H}}$$

is an isomorphism of algebras. Here $SL^{(1)}$ acts on the above tensor product diagonally and it acts on $SL^{(1)}/\mathcal{D}\times\mathcal{H}$ by left multiplication. The mapping Ψ is given by

$$\Psi(f\otimes h)=h(1_{SL^{(1)}})f,$$

where $f\in\mathcal{K}[\mathcal{R}]$, $h\in\mathcal{K}[SL^{(1)}/\mathcal{D}\times\mathcal{H}]$ and $1_{SL^{(1)}}$ is the identity element of the group $SL^{(1)}$. On the other hand,

$$\mathcal{K}[\mathcal{R}]^{\mathcal{D}\times\mathcal{H}}=\mathcal{K}[\mathcal{R}_{\alpha,\phi}(\mathcal{Q},\underline{n})]^{SL_{\alpha,\phi}(\underline{n})}.$$

The action of $\mathcal{G}_1\times\mathcal{G}_2$ on $\mathcal{G}_1\times\mathcal{G}_2/\mathcal{D}$ by left multiplication induces an action of $SL^{(1)}=\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{H}$ on $\mathcal{G}_1\times\mathcal{G}_2/\mathcal{D}$. Thus the $SL^{(1)}$-spaces $SL^{(1)}/\mathcal{D}\times\mathcal{H}$ and $\mathcal{G}_1\times\mathcal{G}_2/\mathcal{D}$ are isomorphic. We define a mapping $\mathcal{G}_1\times\mathcal{G}_2/\mathcal{D}\to SL(n_u)$ by $(g_1,g_2)\to g_1g_2^{-1}$. Since $(g_1,g_2)\cdot g=g_1gg_2^{-1}$ for $g\in SL(n_u)$, $g_1\in\mathcal{G}_1$ and $g_2\in\mathcal{G}_2$, this mapping is an isomorphism of $\mathcal{G}_1\times\mathcal{G}_2$-spaces. Therefore there is an $SL^{(1)}$-equivariant isomorphism of $\mathcal{K}$-algebras

$$\mathcal{K}[SL^{(1)}/\mathcal{D}\times\mathcal{H}]\simeq\mathcal{K}[z_{ij}\,|\,1\le i,j\le n_u]/I,$$

where I is the ideal generated by the polynomial $\det((z_{ij})_{1\le i,j\le n_u})-1$. Consider the natural homomorphism

$$\pi:\mathcal{K}[z_{ij}]\to\mathcal{K}[z_{ij}]/I.$$

Since the $SL^{(1)}$-modules I and $\mathcal{K}[\mathcal{R}]$ have good filtrations (see [8], [7]), the mapping π induces a surjective homomorphism

$$\overline{\pi}:(\mathcal{K}[\mathcal{R}]\otimes\mathcal{K}[z_{ij}])^{SL^{(1)}}\to(\mathcal{K}[\mathcal{R}]\otimes\mathcal{K}[SL^{(1)}/\mathcal{D}\times\mathcal{H}])^{SL^{(1)}}.$$

Moreover, the mapping $\mathcal{K}[\mathcal{R}]\otimes\mathcal{K}[z_{ij}]\to\mathcal{K}[\mathcal{R}_{\alpha^{(1)},\phi^{(1)}}(\mathcal{Q}^{(1)},\underline{n}^{(1)})]$, defined by $z_{ij}\to y_{ij}^b$, is an $SL^{(1)}$-equivariant isomorphism of $\mathcal{K}$-algebras. It remains to observe that the restriction of Φ to semi-invariants is equal to $\Psi\cdot\overline{\pi}$. $\triangle$

5 Semi-invariants of mixed representations of zigzag-quivers

5.1 Notations

Given a zigzag-quiver $\mathcal{Q}=(\mathcal{Q}_0,\mathcal{Q}_1)$, consider its mixed representations of a fixed dimension vector.

Depict $\mathcal{Q}$ together with vector spaces, assigned to vertices, schematically as

$$\begin{array}{cccccc}
\mathcal{V}_1(1)^* & \bullet & & \bullet & \mathcal{V}_1(1) \\
 & \vdots & \xrightarrow{j_+} & \vdots & \\
\mathcal{V}_1(l_1)^* & \bullet & & \bullet & \mathcal{V}_1(l_1) \\
 & & \overset{i}{\nearrow} & & \\
\mathcal{V}_2(1) & \bullet & & \bullet & \mathcal{V}_2(1)^* \\
 & \vdots & \xrightarrow{k_-} & \vdots & \\
\mathcal{V}_2(l_2) & \bullet & & \bullet & \mathcal{V}_2(l_2)^*.
\end{array}$$

Here

- $\mathcal{V}_1(i)$, $\mathcal{V}_2(j)$, $\mathcal{V}_1(i)^*$ and $\mathcal{V}_2(j)^*$ ($1\le i\le l_1$, $1\le j\le l_2$) are vector spaces assigned to vertices. Denote $\dim\mathcal{V}_1(i)=n_i$ and $\dim\mathcal{V}_2(j)=m_j$. Consider $\mathcal{V}_1(i)$, $\mathcal{V}_2(j)$ as spaces of column vectors. Fix the standard bases for $\mathcal{V}_1(i)$, $\mathcal{V}_2(j)$ and the dual bases for $\mathcal{V}_1(i)^*$, $\mathcal{V}_2(j)^*$.

- the arrows of $\mathcal{Q}$ are labeled by symbols i, j_+, k_-, $(1 \le i \le d_1,\ 1 \le j \le d_2,\ 1 \le k \le d_3)$. Introduce notations i', i'', j'_+, j''_+, k'_-, k''_- such that the tail of an arrow i corresponds to the vector space $\mathcal{V}_2(i'')$ and its head corresponds to $\mathcal{V}_1(i')$; the tail of an arrow j_+ corresponds to $\mathcal{V}_1(j''_+)^*$ and its head corresponds to $\mathcal{V}_1(j'_+)$; the tail of an arrow k_- corresponds to $\mathcal{V}_2(k''_-)$ and its head corresponds to $\mathcal{V}_2(k'_-)^*$. Schematically

$$\begin{array}{rcccl} \mathcal{V}_2(i'') & \bullet & \xrightarrow{i} & \bullet & \mathcal{V}_1(i') \\ \mathcal{V}_1(j''_+)^* & \bullet & \xrightarrow{j_+} & \bullet & \mathcal{V}_1(j'_+) \\ \mathcal{V}_2(k''_-) & \bullet & \xrightarrow{k_-} & \bullet & \mathcal{V}_2(k'_-)^*. \end{array}$$

- the involution ϕ permutes vertices of $\mathcal{Q}$ horizontally;
- for a vertex i of zigzag-quiver, $\alpha_i = *$ if and only if the dual vector space is assigned to i.

Denote by $\underline{d}$ the dimension vector, whose entries are n_i, m_j, where $1 \le i \le l_1$, $1 \le j \le l_2$. Then the space of mixed representations of the quiver $\mathcal{Q}$ of dimension $\underline{d}$ is

$$\mathcal{R} = \mathcal{R}_{\alpha,\phi}(\mathcal{Q}, \underline{d}) = \oplus_{i=1}^{d_1} \mathcal{K}^{n_{i'} \times m_{i''}} \bigoplus \oplus_{j=1}^{d_2} \mathcal{K}^{n_{j'_+} \times n_{j''_+}} \bigoplus \oplus_{k=1}^{d_3} \mathrm{Hom}_{\mathcal{K}} \mathcal{K}^{m_{k'_-} \times m_{k''_-}}$$

$$\simeq \oplus_{i=1}^{d_1} \mathrm{Hom}_{\mathcal{K}}(\mathcal{V}_2(i''), \mathcal{V}_1(i')) \bigoplus$$

$$\oplus_{j=1}^{d_2} \mathrm{Hom}_{\mathcal{K}}(\mathcal{V}_1(j''_+)^*, \mathcal{V}_1(j'_+)) \bigoplus \oplus_{k=1}^{d_3} \mathrm{Hom}_{\mathcal{K}}(\mathcal{V}_2(k''_-), \mathcal{V}_2(k'_-)^*),$$

where the isomorphism is given by the choice of bases. Elements of $\mathcal{R}$ are mixed representations of $\mathcal{Q}$ and we write them as $H = (H_1, \ldots, H_{d_1},\ H_1^+, \ldots, H_{d_2}^+,\ H_1^-, \ldots, H_{d_3}^-)$. The group

$$\mathcal{G} = SL_{\alpha,\phi}(\underline{d}) = \prod_{i=1}^{l_1} SL(n_i) \times \prod_{j=1}^{l_2} SL(m_j)$$

acts on $\mathcal{R}$ and on its coordinate ring $\mathcal{K}[\mathcal{R}]$, since $SL(n_i)$ acts on $\mathcal{V}_1(i)$, $\mathcal{V}_1(i)^*$, and $SL(m_j)$ acts on $\mathcal{V}_2(j)$, $\mathcal{V}_2(j)^*$ (see Section 2.2). Denote by

- x_{ij}^k $(1 \le k \le d_1,\ 1 \le i \le n_{k'},\ 1 \le j \le m_{k''})$ the coordinate function on $\mathcal{R}$ that takes a representation H to the (i,j)-th entry of the matrix H_k;

- y^k_{ij} ($1 \le k \le d_2$, $1 \le i \le n_{k'_+}$, $1 \le j \le n_{k''_+}$) the coordinate function on $\mathcal{R}$ that takes H to the (i,j)-th entry of H^+_k;
- z^k_{ij} ($1 \le k \le d_3$, $1 \le i \le m_{k'_-}$, $1 \le j \le m_{k''_-}$) the coordinate function on $\mathcal{R}$ that takes H to the (i,j)-th entry of H^-_k.

Denote by $X_k = (x^k_{ij})$, $Y_k = (y^k_{ij})$, $Z_k = (z^k_{ij})$ the resulting generic matrices.

For $g = (g_1, \ldots, g_{l_1}, h_1, \ldots, h_{l_2}) \in \mathcal{G}$, $1 \le i \le d_1$, $1 \le j \le d_2$, and $1 \le k \le d_3$ we have

$$g \cdot X_i = g_{i'}^{-1} X_i h_{i''},\ g \cdot Y_j = g_{j'_+}^{-1} Y_j (g_{j''_+}^{-1})^T,\ g \cdot Z_k = h_{k'_-}^T Z_k h_{k''_-}.$$

5.2 Formulation of the Theorem

Fix $\underline{t} = (t_1, \ldots, t_{d_1})$, $\underline{r} = (r_1, \ldots, r_{d_2})$, $\underline{s} = (s_1, \ldots, s_{d_3})$ and denote by $\mathcal{K}[\mathcal{R}](\underline{t}, \underline{r}, \underline{s})$ the space of polynomials that have a total degree t_i in variables from X_i ($1 \le i \le d_1$), a total degree r_j in variables from Y_j ($1 \le j \le d_2$), and a total degree s_k in variables from Z_k ($1 \le k \le d_3$). Further, set $t = t_1 + \ldots + t_{d_1}$, $r = r_1 + \ldots + r_{d_2}$, $s = s_1 + \ldots + s_{d_3}$, and let T, R, S be the distributions determined by $\underline{t}, \underline{r}, \underline{s}$, respectively.

Definition 3.(i) A triplet of multidegrees $(\underline{t}, \underline{r}, \underline{s})$ is called *admissible* if there exist $\underline{p} = (p_1, \ldots, p_{l_1}) \in \mathbb{N}^{l_1}$, $\underline{q} = (q_1, \ldots, q_{l_2}) \in \mathbb{N}^{l_2}$ such that

$$\sum_{1 \le k \le d_1,\, k' = i} t_k + \sum_{1 \le k \le d_2,\, k'_+ = i} r_k + \sum_{1 \le k \le d_2,\, k''_+ = i} r_k = n_i p_i \quad (1 \le i \le l_1),$$

$$\sum_{1 \le k \le d_1,\, k'' = i} t_k + \sum_{1 \le k \le d_3,\, k'_- = i} s_k + \sum_{1 \le k \le d_3,\, k''_- = i} s_k = m_i q_i \quad (1 \le i \le l_2).$$

Note that in this case $\sum_{i=1}^{l_1} n_i p_i = t + 2r$ and $\sum_{i=1}^{l_2} m_i q_i = t + 2s$. Set $p = p_1 + \ldots + p_{l_1}$, $q = q_1 + \ldots + q_{l_2}$, and let P, Q be the distributions determined by $\underline{p}, \underline{q}$, respectively. We say that $(\underline{p}, -\underline{q})$ is an *admissible weight*.

(ii) Given an admissible triplet $(\underline{t}, \underline{r}, \underline{s})$, consider a distribution $A = (A_1, \ldots, A_p)$ of the set $[1, t+2r]$ and a distribution $B = (B_1, \ldots, B_q)$ of the set $[1, t+2s]$. If $\#A_i = n_{P|i|}$ ($1 \le i \le p$), $\#B_i = m_{Q|i|}$ ($1 \le i \le q$),

$$\bigcup_{1 \le k \le d_1,\, k' = i} T_k \quad \bigcup_{1 \le k \le d_2,\, k'_+ = i} (t + R_k) \quad \bigcup_{1 \le k \le d_2,\, k''_+ = i} (t + r + R_k) = \bigcup_{1 \le j \le p,\, P|j| = i} A_j \quad (1 \le i \le l_1)$$

and

$$\bigcup_{1\leq k\leq d_1,\, k''=i} T_k \bigcup_{1\leq k\leq d_3,\, k'_-=i} (t{+}S_k) \bigcup_{1\leq k\leq d_3,\, k''_-=i} (t{+}s{+}S_k) = \bigcup_{1\leq j\leq q,\, Q|j|=i} B_j \quad (1\leq i\leq l_2),$$

then we say that the quintuple $(\underline{t},\underline{r},\underline{s},A,B)$ is *admissible.*

Given a distribution $A=(A_1,\ldots,A_p)$ of the set $[1,t+2r]$, define a distribution A' of the set $[1,t]$ and distributions A'',A''' of the set $[1,r]$ as follows:

$$\begin{array}{rl} A'=(A'_1,\ldots,A'_p), & \text{where } A'_j=[1,t]\cap A_j,\\ A''=(A''_1,\ldots,A''_p), & \text{where } A''_j=[1,r]\cap(A_j-t),\\ A'''=(A'''_1,\ldots,A'''_p), & \text{where } A''_j=[1,r]\cap(A_j-t-r). \end{array}$$

Any empty sets should be omitted. In the same way define distributions B',B'',B''' for a distribution $B=(B_1,\ldots,B_q)$ of the set $[1,t+2s]$.

(iii) Let $(\underline{t},\underline{r},\underline{s},A,B)$ be an admissible quintuple.

There are unique multipartitions $\gamma_{max}\vdash\underline{t}$, $\delta_{max}\vdash\underline{r}$, $\lambda_{max}\vdash\underline{s}$ such that the distributions $\Gamma_{max},\Delta_{max},\Lambda_{max}$ determined by $\gamma_{max},\delta_{max},\lambda_{max}$, respectively, satisfy the equalities:

$$\Gamma_{max}=A'\cap B'\cap T,\ \Delta_{max}=A''\cap A'''\cap R,\ \Lambda_{max}=B''\cap B'''\cap S.$$

Obviously, the octuple $(\underline{t},\underline{r},\underline{s},A,B,\gamma_{max},\delta_{max},\lambda_{max})$ is admissible in the following sense:

Consider a multipartition $\gamma\vdash\underline{t}$ of a height $\underline{u}=(u_1,\ldots,u_{d_1})$, a multipartition $\delta\vdash\underline{r}$ of a height $\underline{v}=(v_1,\ldots,v_{d_2})$, and a multipartition $\lambda\vdash\underline{s}$ of a height $\underline{w}=(w_1,\ldots,w_{d_3})$ (the definition of *height* was given in Section 2.3). We set $u=u_1+\ldots+u_{d_1}$, $v=v_1+\ldots+v_{d_2}$, $w=w_1+\ldots+w_{d_3}$, and we write U,V,W, respectively, for the distributions determined by $\underline{u},\underline{v},\underline{w}$, respectively.

If the distributions Γ,Δ,Λ determined by γ,δ,λ, respectively, satisfy

$$\Gamma\leq A'\cap B',\ \Delta\leq A''\cap A''' \text{ and } \Lambda\leq B''\cap B''',$$

then the octuple $(\underline{t},\underline{r},\underline{s},A,B,\gamma,\delta,\lambda)$ is called *admissible.* In other words, such an octuple is admissible if there are mappings

$$a_1:[1,u]\to[1,p],\ a_2,a_3:[1,v]\to[1,p],$$

$$b_1:[1,u]\to[1,q],\ b_2,b_3:[1,w]\to[1,q]$$

such that

$$\Gamma_i \subseteq A_{a_1(i)} \cap B_{b_1(i)},$$
$$t+\Delta_j \subseteq A_{a_2(j)}, \quad t+r+\Delta_j \subseteq A_{a_3(j)},$$
$$t+\Lambda_k \subseteq B_{b_2(k)}, \quad t+s+\Lambda_k \subseteq B_{b_3(k)}$$

for $1 \le i \le u$, $1 \le j \le v$, and $1 \le k \le w$.

Note that for every γ, δ, λ such that $(\underline{t}, \underline{r}, \underline{s}, A, B, \gamma, \delta, \lambda)$ is admissible we have $\gamma \le \gamma_{max}$, $\delta \le \delta_{max}$, $\lambda \le \lambda_{max}$.

Construction 2. Let $(\underline{t}, \underline{r}, \underline{s}, A, B, \gamma, \delta, \lambda)$ be an admissible octuple.

(i) Given $1 \le k \le u$, define a $(t+2r) \times (t+2s)$ matrix $[X]_k$, partitioned into $p \times q$ number of blocks, where the block in the (i,j)-th position is an $n_{P|i|} \times m_{Q|j|}$ matrix; the block in the $(a_1(k), b_1(k))$-th position is equal to $X_{U|k|}$, and the rest of blocks are zero matrices.

Given $1 \le k \le v$, define a $(t+2r) \times (t+2r)$ matrix $[Y]_k$, partitioned into $p \times p$ number of blocks, where the block in the (i,j)-th position is an $n_{P|i|} \times n_{P|j|}$ matrix; the block in the $(a_2(k), a_3(k))$-th position is equal to $Y_{V|k|}$, and the rest of blocks are zero matrices.

Given $1 \le k \le w$, define a $(t+2s) \times (t+2s)$ matrix $[Z]_k$, partitioned into $q \times q$ number of blocks, where the block in the (i,j)-th position is a $m_{Q|i|} \times m_{Q|j|}$ matrix; the block in the $(b_2(k), b_3(k))$-th position is equal to $Z_{W|k|}$, and the rest of blocks are zero matrices.

(ii) For $(\underline{t}, \underline{r}, \underline{s}, A, B)$ consider γ_{max},δ_{max},λ_{max}, defined in part (iii) of Definition 3. We set

$$\mathrm{DP}^{A,B}_{\gamma,\delta,\lambda} = \mathrm{DP}_{\gamma,\delta,\lambda}([X]_1, \ldots, [X]_u, [Y]_1, \ldots, [Y]_v, [Z]_1, \ldots, [Z]_w)$$

and

$$\mathrm{DP}^{A,B}_{\underline{t},\underline{r},\underline{s}} = \mathrm{DP}^{A,B}_{\gamma_{max},\delta_{max},\lambda_{max}}.$$

We will see in Lemma 7 that $\mathrm{DP}^{A,B}_{\underline{t},\underline{r},\underline{s}}$ is well defined.

Theorem 2 *If a space $\mathcal{K}[\mathcal{R}](\underline{t}, \underline{r}, \underline{s})^{\mathcal{G}}$ is non-zero, then the triplet $(\underline{t}, \underline{r}, \underline{s})$ is admissible. In this case $\mathcal{K}[\mathcal{R}](\underline{t}, \underline{r}, \underline{s})^{\mathcal{G}}$ is spanned over $\mathcal{K}$ by the set of all $\mathrm{DP}^{A,B}_{\underline{t},\underline{r},\underline{s}}$ for admissible quintuples $(\underline{t}, \underline{r}, \underline{s}, A, B)$.*

Note that in the characteristic zero case Proposition 3 (see below) provides a formula for calculation of $\mathrm{DP}_{\underline{t},\underline{r},\underline{s}}^{A,B}$, without using functions $a_1, a_2, a_3, b_1, b_2, b_3$.

Theorem 2 enables us to describe relative invariants (for definition see Section 2.2).

Corollary 1 *All relative invariants in $\mathcal{K}[\mathcal{R}]$ have admissible weights. The space of relative invariants of the admissible weight $(\underline{p}, -\underline{q})$ is spanned over $\mathcal{K}$ by the set of all $\mathrm{DP}_{\underline{t},\underline{r},\underline{s}}^{A,B}$ for admissible quintuples $(\underline{t}, \underline{r}, \underline{s}, A, B)$ with weight $(\underline{p}, -\underline{q})$.*

Proof. See below the proof of Lemma 7. $\triangle$

5.3 Particular Case of $l_1 = l_2 = 1$

To illustrate Theorem 2, we consider a simple particular case which contains all the main features of the general case. Also, during the first reading of the proof of Theorem 2 it may be useful to work in this particular case; it is quite easy to rewrite the proof for this case, using the following remarks.

Assume $l_1 = l_2 = 1$. We set $\dim \mathcal{V}_1(1) = n$, $\dim \mathcal{V}_2(1) = m$. Then Definition 3 and Construction 2 turn into:

Definition 3'. (i) A triplet of multidegrees $(\underline{t}, \underline{r}, \underline{s})$ is called *admissible* if there are $p, q \in \mathbb{N}$ such that

$$t + 2r = np, \ t + 2s = mq.$$

(ii) Let $(\underline{t}, \underline{r}, \underline{s})$ be an admissible triplet and $A = (A_1, \ldots, A_p)$, $B = (B_1, \ldots, B_q)$, respectively, be distributions of the sets $[1, t + 2r]$ and $[1, t + 2s]$, respectively. Then the quintuple $(\underline{t}, \underline{r}, \underline{s}, A, B)$ is *admissible* if $\#A_i = n$ and $\#B_j = m$ for $1 \leq i \leq p$, $1 \leq j \leq q$.

(iii) See item (iii) from Definition 3.

Construction 2'. See Construction 2. Note that sizes of all blocks of matrices $[X]_k$ ($[Y]_k$, $[Z]_k$, respectively) are $n \times m$ ($n \times n$, $m \times m$, respectively).

In this case it is not necessary to assume that a quintuple $(\underline{t}, \underline{r}, \underline{s}, A, B)$ is admissible, hence we could simplify the result by getting rid of the distributions A, B:

Proposition 1 *Let $l_1 = l_2 = 1$. If a triplet $(\underline{t}, \underline{r}, \underline{s})$ is admissible, then $\mathcal{K}[\mathcal{R}](\underline{t}, \underline{r}, \underline{s})^{\mathcal{G}}$ is spanned over $\mathcal{K}$ by the set of all*

$$\mathrm{DP}_{\gamma,\delta,\lambda}([X]_1, \ldots, [X]_u, [Y]_1, \ldots, [Y]_v, [Z]_1, \ldots, [Z]_w),$$

for arbitrary multipartitions $\gamma \vdash \underline{t}$, $\delta \vdash \underline{r}$, $\lambda \vdash \underline{s}$ *of heights* $\underline{u}$, $\underline{v}$, $\underline{w}$, *respectively, and arbitrary mappings*

$$a_1 : [1,u] \to [1,p],\ a_2, a_3 : [1,v] \to [1,p],$$

$$b_1 : [1,u] \to [1,q],\ b_2, b_3 : [1,w] \to [1,q],$$

which define $[X]_i$, $[Y]_j$, $[Z]_k$ *(see Construction* 2*).*

Proof. The proposition follows immediately from Theorem 2 and the proof of Lemma 7 (see below). Note that it is obvious that matrices $[X]_i$, $[Y]_j$, $[Z]_k$ are well defined. $\triangle$

6 Proof of Theorem 2

Lemma 7 *For an admissible octuple* $(\underline{t}, \underline{r}, \underline{s}, A, B, \gamma, \delta, \lambda)$ *the element* $\mathrm{DP}^{A,B}_{\gamma,\delta,\lambda}$ *is well defined and belongs to* $\mathcal{K}[\mathcal{R}](\underline{t}, \underline{r}, \underline{s})^{\mathcal{G}}$.

Proof. The first part of the lemma follows by straightforward calculations.

Given $1 \le k \le u$ and an $n_{P|a_1(k)|} \times m_{Q|b_1(k)|}$ matrix X, substitute X for the only non-zero block in $[X]_k$ and denote the resulting $(t+2r) \times (t+2s)$ matrix by $[X]^{(1)}_k$. In the same manner for $1 \le k \le v$ and an $n_{P|a_2(k)|} \times n_{P|a_3(k)|}$ matrix Y define $(t+2r) \times (t+2r)$ matrix $[Y]^{(2)}_k$, and for $1 \le k \le w$ and a $m_{Q|b_2(k)|} \times m_{Q|b_3(k)|}$ matrix Z define $(t+2s) \times (t+2s)$ matrix $[Z]^{(3)}_k$.

For $g = (g_1, \ldots, g_{l_1}, h_1, \ldots, h_{l_2}) \in \mathcal{G}$ define block-diagonal matrices g_0, h_0 of sizes $(t+2r) \times (t+2r)$ and $(t+2s) \times (t+2s)$, respectively, in such a way that the i-th block of the matrix g_0 is equal to $g_{P|i|}$ $(1 \le i \le p)$ and the j-th block of the matrix h_0 is equal to $h_{Q|j|}$ $(1 \le j \le q)$. For variables $x_1, \ldots, x_u, y_1, \ldots, y_v, z_1, \ldots, z_w$ we have

$$g \cdot \mathrm{DP}_{r,s}(\sum_{i=1}^{u} x_i [X_{U|i|}]^{(1)}_i, \sum_{j=1}^{v} y_j [Y_{V|j|}]^{(2)}_j, \sum_{k=1}^{w} z_k [Z_{W|k|}]^{(3)}_k)$$

$$= \mathrm{DP}_{r,s}(\sum_{i=1}^{u} x_i [g^{-1}_{U|i|'} X_{U|i|}\, h_{U|i|''}]^{(1)}_i,$$

$$\sum_{j=1}^{v} y_j [g^{-1}_{V|j|'_+} Y_{V|j|}\, (g^{-1}_{V|j|''_+})^T]^{(2)}_j, \sum_{k=1}^{w} z_k [h^T_{W|k|'_-} Z_{W|k|}\, h_{W|k|''_-}]^{(3)}_k)$$

$$= \mathrm{DP}_{r,s}(g_0^{-1}\sum_{i=1}^{u} x_i[X_{U|i|}]_i^{(1)}h_0, g_0^{-1}\sum_{j=1}^{v} y_j[Y_{V|j|}]_j^{(2)}(g_0^{-1})^T, h_0^T\sum_{k=1}^{w} z_k[Z_{W|k|}]_k^{(3)}h_0)$$

which by Lemma 4 equals

$$\det(g_0)\det(h_0)^{-1}\mathrm{DP}_{r,s}(\sum_{i=1}^{u} x_i[X_{U|i|}]_i^{(1)}, \sum_{j=1}^{v} y_j[Y_{V|j|}]_j^{(2)}, \sum_{k=1}^{w} z_k[Z_{W|k|}]_k^{(3)})$$

and the statement follows. $\triangle$

We set

$$M_1(\underline{t},\underline{r},\underline{s}) = \otimes_{i=1}^{d_1} S^{t_i}(\mathcal{V}_1(i')^*\otimes\mathcal{V}_2(i''))\bigotimes$$
$$\otimes_{j=1}^{d_2} S^{r_j}(\mathcal{V}_1(j'_+)^*\otimes\mathcal{V}_1(j''_+)^*)\bigotimes\otimes_{k=1}^{d_3} S^{s_k}(\mathcal{V}_2(k'_-)\otimes\mathcal{V}_2(k''_-)).$$

By Lemma 1, there is an isomorphism of $\mathcal{G}$-modules $\Phi:\mathcal{K}[\mathcal{R}](\underline{t},\underline{r},\underline{s})\to M_1(\underline{t},\underline{r},\underline{s})$. Put

$$M_2(\underline{t},\underline{r},\underline{s},\gamma,\delta,\lambda) = \otimes_{i=1}^{d_1}(\wedge^{\gamma_i}\mathcal{V}_1(i')^*\otimes\wedge^{\gamma_i}\mathcal{V}_2(i''))\bigotimes$$
$$\otimes_{j=1}^{d_2}(\wedge^{\delta_j}\mathcal{V}_1(j'_+)^*\otimes\wedge^{\delta_j}\mathcal{V}_1(j''_+)^*)\bigotimes\otimes_{k=1}^{d_3}(\wedge^{\lambda_k}\mathcal{V}_2(k'_-)\otimes\wedge^{\lambda_k}\mathcal{V}_2(k''_-))$$

and define a mapping

$$\zeta_{\gamma,\delta,\lambda} = \otimes_{i=1}^{d_1}\zeta_{\gamma_i}\bigotimes\otimes_{j=1}^{d_2}\zeta_{\delta_j}\bigotimes\otimes_{k=1}^{d_3}\zeta_{\lambda_k}: M_2(\underline{t},\underline{r},\underline{s},\gamma,\delta,\lambda)\to M_1(\underline{t},\underline{r},\underline{s}).$$

Lemma 8 *Given multidegrees $\underline{t},\underline{r},\underline{s}$, we have*

$$\mathcal{K}[\mathcal{R}](\underline{t},\underline{r},\underline{s})^{\mathcal{G}} = \sum_{\gamma\vdash\underline{t}}\sum_{\delta\vdash\underline{r}}\sum_{\lambda\vdash\underline{s}}\Phi^{-1}\zeta_{\gamma,\delta,\lambda}(M_2(\underline{t},\underline{r},\underline{s},\gamma,\delta,\lambda)^{\mathcal{G}}).$$

Proof. Word-by-word repeat the proof of Proposition 5.1 from [7]. The only difference is that we consider $M_1(\underline{t},\underline{r},\underline{s})$ instead of another tensor product from the mentioned Proposition. $\triangle$

Lemma 9 *([7], Proposition 5.2) Let $\gamma=(\gamma_1,\ldots,\gamma_u)\in\mathbb{N}^u$, $\gamma_1+\ldots+\gamma_u=t$, and let $\mathcal{V}$ be a vector space with a basis $e_1,\ldots,e_n$. If $(\wedge^{\gamma}\mathcal{V})^{SL(\mathcal{V})}\neq 0$, then $t=np$ for some $p\in\mathbb{N}$. In this case $(\wedge^{\gamma}\mathcal{V})^{SL(\mathcal{V})}$ is spanned over $\mathcal{K}$ by the elements $\sum_{\tau\in\mathcal{S}_C/\mathcal{S}_C\cap\mathcal{S}_\Gamma}\mathrm{sgn}(\tau)\otimes_{i=1}^{u}\wedge_{k\in\Gamma_i}e_{C\langle\tau(k)\rangle}$, where $C=(C_1,\ldots,C_p)$ is a distribution of $[1,t]$, $\#C_i=n$ for any $1\le i\le p$.*

Let a quintuple $(\underline{t}, \underline{r}, \underline{s}, A, B)$ be admissible. Given $\rho_1 \in \mathcal{S}_{t+2r}$ and $\rho_2 \in \mathcal{S}_{t+2s}$ put

$$F^{A,B}(\rho_1, \rho_2) = \mathrm{sgn}(\rho_1\rho_2)\cdot$$

$$\prod_{i=1}^{t} x^{T|i|}_{A\langle\rho_1(i)\rangle, B\langle\rho_2(i)\rangle} \prod_{j=1}^{r} y^{R|j|}_{A\langle\rho_1(t+j)\rangle, A\langle\rho_1(t+r+j)\rangle} \prod_{k=1}^{s} z^{S|k|}_{B\langle\rho_2(t+k)\rangle, B\langle\rho_2(t+s+k)\rangle}. \qquad (4)$$

The following lemma follows immediately by index change in products.

Lemma 10 *Let $\rho_1 \in \mathcal{S}_{t+2r}$ and $\rho_2 \in \mathcal{S}_{t+2s}$. Then*

1) $F^{A,B}(\rho_1 \cdot (\pi \times id_r \times id_r), \rho_2) = F^{A,B}(\rho_1, \rho_2 \cdot (\pi^{-1} \times id_s \times id_s))$ for $\pi \in \mathcal{S}_T$.

2) $F^{A,B}(\rho_1 \cdot (id_t \times \pi \times id_r), \rho_2) = F^{A,B}(\rho_1 \cdot (id_t \times id_r \times \pi^{-1}), \rho_2)$ for $\pi \in \mathcal{S}_R$.

3) $F^{A,B}(\rho_1, \rho_2 \cdot (id_t \times \pi \times id_s)) = F^{A,B}(\rho_1, \rho_2 \cdot (id_t \times id_s \times \pi^{-1}))$ for $\pi \in \mathcal{S}_S$.

Given an admissible quintuple $(\underline{t}, \underline{r}, \underline{s}, A, B)$ and distributions Γ, Δ, Λ, respectively, determined by multipartitions $\gamma \vdash \underline{t}$, $\delta \vdash \underline{r}$, $\lambda \vdash \underline{s}$, respectively, define

$$H^{A,B}_{\gamma,\delta,\lambda} = \sum_{\tau_1 \in \mathcal{S}_A/\mathcal{L}_1} \sum_{\tau_2 \in \mathcal{S}_B/\mathcal{L}_2} \sum_{\sigma_1 \in \mathcal{S}_\Gamma} \sum_{\sigma_2 \in \mathcal{S}_\Delta} \sum_{\sigma_3 \in \mathcal{S}_\Lambda} F^{A,B}(\tau_1 \cdot (id_t \times id_r \times \sigma_2), \tau_2 \cdot (\sigma_1 \times id_s \times \sigma_3)),$$

where $\mathcal{L}_1 = \mathcal{S}_A \cap (\mathcal{S}_\Gamma \times \mathcal{S}_\Delta \times \mathcal{S}_\Delta)$, $\mathcal{L}_2 = \mathcal{S}_B \cap (\mathcal{S}_\Gamma \times \mathcal{S}_\Lambda \times \mathcal{S}_\Lambda)$.

Proposition 2 *1. If a triplet $(\underline{t}, \underline{r}, \underline{s})$ is not admissible, then $\mathcal{K}[\mathcal{R}](\underline{t}, \underline{r}, \underline{s})^{\mathcal{G}} = 0$.*

2. If a triplet $(\underline{t}, \underline{r}, \underline{s})$ is admissible, then $\mathcal{K}[\mathcal{R}](\underline{t}, \underline{r}, \underline{s})^{\mathcal{G}}$ is spanned over $\mathcal{K}$ by the set $\{H^{A,B}_{\gamma,\delta,\lambda} \mid$ a quintuple $(\underline{t}, \underline{r}, \underline{s}, A, B)$ is admissible, $\gamma \vdash \underline{t}$, $\delta \vdash \underline{r}$, $\lambda \vdash \underline{s}\}$.

Proof. Rewrite $M_2(\underline{t}, \underline{r}, \underline{s}, \gamma, \delta, \lambda)^{\mathcal{G}}$ as

$$\otimes_{i=1}^{l_1} \Big(\bigotimes_{\substack{0<k\le d_1\\ k'=i}} \wedge^{\gamma_k} \mathcal{V}_1(k')^* \bigotimes \bigotimes_{\substack{0<k\le d_2\\ k'_+=i}} \wedge^{\delta_k} \mathcal{V}_1(k'_+)^* \bigotimes \bigotimes_{\substack{0<k\le d_2\\ k''_+=i}} \wedge^{\delta_k} \mathcal{V}_1(k''_+)^* \Big)^{SL(n_i)} \bigotimes$$

$$\otimes_{j=1}^{l_2} \Big(\bigotimes_{\substack{0<k\le d_1\\ k''=j}} \wedge^{\gamma_k} \mathcal{V}_2(k'') \bigotimes \bigotimes_{\substack{0<k\le d_3\\ k'_-=j}} \wedge^{\lambda_k} \mathcal{V}_2(k'_-) \bigotimes \bigotimes_{\substack{0<k\le d_3\\ k''_-=j}} \wedge^{\lambda_k} \mathcal{V}_2(k''_-) \Big)^{SL(m_j)}$$

and use Lemmas 8 and 9. $\triangle$

Remark 2 *From now on we work over the field $\mathbb{Q}$ instead of a field $\mathcal{K}$. The case of positive characteristic is considered at the end of the section (see Remark 3). We will use formulas for $\mathrm{DP}^{A,B}_{\gamma,\delta,\lambda}$ and $H^{A,B}_{\gamma,\delta,\lambda}$ with denominators, i.e.*

$$H^{A,B}_{\gamma,\delta,\lambda} = \frac{1}{c} \sum_{\tau_1 \in \mathcal{S}_A} \sum_{\tau_2 \in \mathcal{S}_B} \sum_{\sigma_1 \in \mathcal{S}_\Gamma} \sum_{\sigma_2 \in \mathcal{S}_\Delta} \sum_{\sigma_3 \in \mathcal{S}_\Lambda} F^{A,B}(\tau_1 \cdot (id_t, id_r, \sigma_2), \tau_2 \cdot (\sigma_1, id_s, \sigma_3)),$$

where $c = \#(\mathcal{S}_A \cap (\mathcal{S}_\Gamma \times \mathcal{S}_\Delta \times \mathcal{S}_\Delta))\,\#(\mathcal{S}_B \cap (\mathcal{S}_\Gamma \times \mathcal{S}_\Lambda \times \mathcal{S}_\Lambda))$, and similarly for $\mathrm{DP}^{A,B}_{\gamma,\delta,\lambda}$.

Proposition 3 *Assume an octuple $(\underline{t},\underline{r},\underline{s},A,B,\gamma,\delta,\lambda)$ is admissible. Consider permutations $\pi_1\in\mathcal{S}_{t+2r}$, $\pi_2\in\mathcal{S}_{t+2s}$ such that $A=N^{\pi_1}$, $B=M^{\pi_2}$ and $A\langle i\rangle=N\langle\pi_1(i)\rangle$, $B\langle j\rangle=M\langle\pi_2(j)\rangle$ for $1\leq i\leq t+2r$, $1\leq j\leq t+2s$, where N,M are distributions determined by the vectors $\underline{n}=(n_1,\ldots,n_1,\ldots,n_{l_1},\ldots,n_{l_1})\in\mathbb{N}^{p_1+\ldots+p_{l_1}}$, $\underline{m}=(m_1,\ldots,m_1,\ldots,m_{l_2},\ldots,m_{l_2})\in\mathbb{N}^{q_1+\ldots+q_{l_2}}$, respectively. Then*

$$\mathrm{DP}^{A,B}_{\gamma,\delta,\lambda}=\mathrm{sgn}(\pi_1\pi_2)\frac{1}{\#\mathcal{S}_\Gamma\,\#\mathcal{S}_\Delta\,\#\mathcal{S}_\Lambda}\sum_{\tau_1\in\mathcal{S}_A}\sum_{\tau_2\in\mathcal{S}_B}F^{A,B}(\tau_1,\tau_2),$$

where $F^{A,B}$ is defined by formula (4).

Proof. Note that the admissibility of the octuple implies $(\mathcal{S}_\Gamma\times\mathcal{S}_\Delta\times\mathcal{S}_\Delta)\subseteq\mathcal{S}_A$, $(\mathcal{S}_\Gamma\times\mathcal{S}_\Lambda\times\mathcal{S}_\Lambda)\subseteq\mathcal{S}_B$.

Define $[X]_k=(f^k_{ij})$, $[Y]_k=(g^k_{ij})$ and $[Z]_k=(h^k_{ij})$. By Lemma 5 and the definition

$$\mathrm{DP}^{A,B}_{\gamma,\delta,\lambda}=\frac{1}{\#\mathcal{S}_\Gamma\,\#\mathcal{S}_\Delta\,\#\mathcal{S}_\Lambda}\sum_{\tau_1\in\mathcal{S}_{t+2r}}\sum_{\tau_2\in\mathcal{S}_{t+2s}}\mathrm{sgn}(\tau_1\tau_2)L,\ \text{where}$$

$$L=\prod_{i=1}^{t}f^{\Gamma|i|}_{\tau_1(i),\tau_2(i)}\prod_{j=1}^{r}g^{\Delta|j|}_{\tau_1(t+j),\tau_1(t+r+j)}\prod_{k=1}^{s}h^{\Lambda|k|}_{\tau_2(t+k),\tau_2(t+s+k)}.$$

For any $1\leq i\leq t$, $1\leq j\leq r$, and $1\leq k\leq s$ we have

$$f^{\Gamma|i|}_{\tau_1(i),\tau_2(i)}=\left\{\begin{array}{cl} x^{U|\Gamma|i||}_{N\langle\tau_1(i)\rangle,M\langle\tau_2(i)\rangle}, & \text{if } \tau_1(i)\in N_{a_1(\Gamma|i|)} \text{ and}\\ & \quad \tau_2(i)\in M_{b_1(\Gamma|i|)}\\ 0, & \text{otherwise ,}\end{array}\right.$$

$$g^{\Delta|j|}_{\tau_1(t+j),\tau_1(t+r+j)}=\left\{\begin{array}{cl} y^{V|\Delta|j||}_{N\langle\tau_1(t+j)\rangle,N\langle\tau_1(t+r+j)\rangle}, & \text{if } \tau_1(t+j)\in N_{a_2(\Delta|j|)} \text{ and}\\ & \quad \tau_1(t+r+j)\in N_{a_3(\Delta|j|)}\\ 0, & \text{otherwise ,}\end{array}\right.$$

$$h^{\Lambda|k|}_{\tau_2(t+k),\tau_2(t+s+k)}=\left\{\begin{array}{cl} z^{W|\Lambda|k||}_{M\langle\tau_2(t+k)\rangle,M\langle\tau_2(t+s+k)\rangle}, & \text{if } \tau_2(t+k)\in M_{b_2(\Lambda|k|)} \text{ and}\\ & \quad \tau_2(t+s+k)\in M_{b_3(\Lambda|k|)}\\ 0, & \text{otherwise .}\end{array}\right.$$

Since the octuple is admissible, it implies $i\in N^{\pi_1}_{a_1(\Gamma|i|)}\cap M^{\pi_2}_{b_1(\Gamma|i|)}$,

$$t+j\in N^{\pi_1}_{a_2(\Delta|j|)},\ t+r+j\in N^{\pi_1}_{a_3(\Delta|j|)},$$

$$t+k\in M^{\pi_2}_{b_2(\Lambda|k|)},\ t+s+k\in M^{\pi_2}_{b_3(\Lambda|k|)}.$$

Therefore

$$\pi_1(i) \in N_{a_1(\Gamma|i|)},\ \pi_1(t+j) \in N_{a_2(\Delta|j|)},\ \pi_1(t+r+j) \in N_{a_3(\Delta|j|)},$$

$$\pi_2(i) \in M_{b_1(\Gamma|i|)},\ \pi_2(t+k) \in M_{b_2(\Lambda|k|)},\ \pi_2(t+s+k) \in M_{b_3(\Lambda|k|)}.$$

If $\tau_1 \in \mathcal{S}_N\pi_1$ and $\tau_2 \in \mathcal{S}_M\pi_2$, then L equals

$$\prod_{i=1}^{t} x^{T|i|}_{N\langle\tau_1(i)\rangle,M\langle\tau_2(i)\rangle} \prod_{j=1}^{r} y^{R|j|}_{N\langle\tau_1(t+j)\rangle,N\langle\tau_1(t+r+j)\rangle} \prod_{k=1}^{s} z^{S|k|}_{M\langle\tau_2(t+k)\rangle,M\langle\tau_2(t+s+k)\rangle};$$

otherwise L is equal to zero. Use a substitution $\tau_1 \to \pi_1\sigma_1$, $\tau_2 \to \pi_2\sigma_2$ in the given expression for $\mathrm{DP}^{A,B}_{\gamma,\delta,\underline{\gamma}}$ to conclude the proof. $\triangle$

Corollary 2 *Suppose an octuple $(\underline{t},\underline{r},\underline{s},A,B,\gamma,\delta,\lambda)$ is admissible and γ_{max}, δ_{max}, λ_{max} corresponds to $(\underline{t},\underline{r},\underline{s},A,B)$. Then*

$$\mathrm{DP}^{A,B}_{\gamma,\delta,\lambda} = \pm c\,\mathrm{DP}^{A,B}_{\gamma_{max},\delta_{max},\lambda_{max}},$$

where $c = \#\mathcal{S}_{\Gamma_{max}}\#\mathcal{S}_{\Delta_{max}}\#\mathcal{S}_{\Lambda_{max}}\Big/\#\mathcal{S}_{\Gamma}\,\#\mathcal{S}_{\Delta}\,\#\mathcal{S}_{\Lambda} \in \mathbb{N}$.

Lemma 11 *Assume a quintuple $(\underline{t},\underline{r},\underline{s},A,B)$ is admissible, $\gamma \vdash \underline{t}$, $\delta \vdash \underline{r}$, and $\lambda \vdash \underline{s}$. Then there are $\gamma_0 \vdash \underline{t}$, $\delta_0 \vdash \underline{r}$, $\lambda_0 \vdash \underline{s}$, $\rho_1 \in \mathcal{S}_\Gamma$, $\rho_2 \in \mathcal{S}_\Delta$, and $\rho_3 \in \mathcal{S}_\Lambda$ such that the octuple $(\underline{t},\underline{r},\underline{s},A^{\rho_1\times\rho_2\times\rho_2},B^{\rho_1\times\rho_3\times\rho_3},\gamma_0,\delta_0,\lambda_0)$ is admissible, $\mathcal{S}_{\Gamma_0} = \mathcal{S}^{\rho_1}_{A'} \cap \mathcal{S}_\Gamma \cap \mathcal{S}^{\rho_2}_{B'}$, $\mathcal{S}_{\Delta_0} = \mathcal{S}^{\rho_2}_{A''} \cap \mathcal{S}_\Delta \cap \mathcal{S}^{\rho_2}_{A'''}$, and $\mathcal{S}_{\Lambda_0} = \mathcal{S}^{\rho_3}_{B''} \cap \mathcal{S}_\Lambda \cap \mathcal{S}^{\rho_3}_{B'''}$, where $\Gamma,\Delta,\Lambda,\Gamma_0,\Delta_0,\Lambda_0$ are the distributions determined by $\gamma,\delta,\lambda,\gamma_0,\delta_0,\lambda_0$, respectively.*

Proof. Intersecting the distributions A',Γ,B' of the set $[1,t]$, we obtain the distribution $\Gamma_0^{\rho_1^{-1}}$ for some $\gamma_0 \vdash \underline{t}$, $\rho_1 \in \mathcal{S}_\Gamma$. Intersecting the distributions A'',Δ,A''' of the set $[1,r]$, we obtain the distribution $\Delta_0^{\rho_2^{-1}}$ for some $\delta_0 \vdash \underline{r}$, $\rho_2 \in \mathcal{S}_\Delta$. Intersecting the distributions B'',Λ,B''' of the set $[1,s]$, we obtain the distribution $\Lambda_0^{\rho_3^{-1}}$ for some $\lambda_0 \vdash \underline{s}$, $\rho_3 \in \mathcal{S}_\Lambda$. Such elements $\gamma_0,\delta_0,\lambda_0,\rho_1,\rho_2,\rho_3$ satisfy the required properties. $\triangle$

Proposition 4 *Assume that a quintuple $(\underline{t},\underline{r},\underline{s},A,B)$ is admissible, $\gamma\vdash\underline{t}$, $\delta\vdash\underline{r}$, $\lambda\vdash\underline{s}$, and Γ,Δ,Λ are distributions determined by γ,δ,λ, respectively. Then the semi-invariant $H^{A,B}_{\gamma,\delta,\lambda}$ is equal to*

$$\sum_{\pi_1^{-1}\in\mathcal{S}_\Gamma\cap\mathcal{S}_{A'}\backslash\mathcal{S}_\Gamma/\mathcal{S}_\Gamma\cap\mathcal{S}_{B'}}\;\sum_{\pi_2^{-1}\in\mathcal{S}_\Delta\cap\mathcal{S}_{A''}\backslash\mathcal{S}_\Delta/\mathcal{S}_\Delta\cap\mathcal{S}_{A'''}}\;\sum_{\pi_3^{-1}\in\mathcal{S}_\Lambda\cap\mathcal{S}_{B''}\backslash\mathcal{S}_\Lambda/\mathcal{S}_\Lambda\cap\mathcal{S}_{B'''}}\pm\mathrm{DP}^{A_0,B_0}_{\gamma_0,\delta_0,\lambda_0},$$

where $\gamma_0\vdash\underline{t}$, $\delta_0\vdash\underline{r}$, $\lambda_0\vdash\underline{s}$, $A_0=A^{\rho_1\times\rho_2\times\pi_2\rho_2}$, $B_0=B^{\pi_1\rho_1\times\rho_3\times\pi_3\rho_3}$, and $\rho_1\in\mathcal{S}_t$, $\rho_2\in\mathcal{S}_r$, $\rho_3\in\mathcal{S}_s$ depend on $\pi_1,\pi_2,\pi_3,\underline{t},\underline{r},\underline{s},A,B,\gamma,\delta,\lambda$. Moreover, the octuple $(\underline{t},\underline{r},\underline{s},A^{\rho_1\times\rho_2\times\pi_2\rho_2},B^{\pi_1\rho_1\times\rho_3\times\pi_3\rho_3},\gamma_0,\delta_0,\lambda_0)$ is admissible.

Proof. Since $\mathcal{S}_\Gamma\subseteq\mathcal{S}_T$, $\mathcal{S}_\Delta\subseteq\mathcal{S}_R$ and $\mathcal{S}_\Lambda\subseteq\mathcal{S}_S$ we can use Lemma 10 throughout the proof. Denote

$$c_1=\#((\mathcal{S}_\Gamma\times\mathcal{S}_\Delta\times\mathcal{S}_\Delta)\cap\mathcal{S}_A)\;\#((\mathcal{S}_\Gamma\times\mathcal{S}_\Lambda\times\mathcal{S}_\Lambda)\cap\mathcal{S}_B)$$

$$=\#(\mathcal{S}_\Gamma\cap\mathcal{S}_{A'})\cdot\#(\mathcal{S}_\Delta\cap\mathcal{S}_{A''})\cdot\#(\mathcal{S}_\Delta\cap\mathcal{S}_{A'''})\cdot\#(\mathcal{S}_\Gamma\cap\mathcal{S}_{B'})\cdot\#(\mathcal{S}_\Lambda\cap\mathcal{S}_{B''})\cdot\#(\mathcal{S}_\Lambda\cap\mathcal{S}_{B'''}).$$

For $1\leq i\leq 3$ substitute σ_i^{-1} for σ_i in the definition of $H^{A,B}_{\gamma,\delta,\lambda}$ to get

$$H^{A,B}_{\gamma,\delta,\lambda}=\frac{1}{c_1}\sum_{\tau_1\in\mathcal{S}_A,\tau_2\in\mathcal{S}_B}\sum_{\sigma_1\in\mathcal{S}_\Gamma}\sum_{\sigma_2\in\mathcal{S}_\Delta}\sum_{\sigma_3\in\mathcal{S}_\Lambda}F^{A,B}(\tau_1\cdot(id_t\times id_r\times\sigma_2^{-1}),\tau_2\cdot(\sigma_1^{-1}\times id_s\times\sigma_3^{-1})).$$

Define $\mathcal{M}_1=\mathcal{S}_\Gamma/\mathcal{S}_\Gamma\cap\mathcal{S}_{B'}$, $\mathcal{M}_2=\mathcal{S}_\Delta/\mathcal{S}_\Delta\cap\mathcal{S}_{A'''}$, $\mathcal{M}_3=\mathcal{S}_\Lambda/\mathcal{S}_\Lambda\cap\mathcal{S}_{B'''}$,

$$\mathcal{L}_1=\mathcal{S}_\Gamma\cap\mathcal{S}_{B'},\ \mathcal{L}_2=\mathcal{S}_\Delta\cap\mathcal{S}_{A'''},\ \mathcal{L}_3=\mathcal{S}_\Lambda\cap\mathcal{S}_{B'''}.$$

Then $H^{A,B}_{\gamma,\delta,\lambda}$

$$=\frac{1}{c_1}\sum_{\substack{\sigma_i\in\mathcal{M}_i\\1\leq i\leq 3}}\sum_{\substack{\nu_i\in\mathcal{L}_i\\1\leq i\leq 3}}\sum_{\substack{\tau_1\in\mathcal{S}_A\\\tau_2\in\mathcal{S}_B}}F^{A,B}(\tau_1\cdot(id_t\times id_r\times\nu_2^{-1}\sigma_2^{-1}),\tau_2\cdot(\nu_1^{-1}\sigma_1^{-1}\times id_s\times\nu_3^{-1}\sigma_3^{-1}))$$

which is equal to

$$\frac{1}{c_2}\sum_{\sigma_i\in\mathcal{M}_i,1\leq i\leq 3}\sum_{\tau_1\in\mathcal{S}_A,\tau_2\in\mathcal{S}_B}F^{A,B}(\tau_1\cdot(\sigma_1\times\sigma_2\times id_r),\tau_2\cdot(id_t\times\sigma_3\times id_s)),$$

where $c_2=\#(\mathcal{S}_\Gamma\cap\mathcal{S}_{A'})\,\#(\mathcal{S}_\Delta\cap\mathcal{S}_{A''})\,\#(\mathcal{S}_\Lambda\cap\mathcal{S}_{B''})$, after we substituted $\tau_1\cdot(id_t\times id_r\times\nu_2^{-1})\to\tau_1$ and $\tau_2\cdot(\nu_1^{-1}\times id_s\times\nu_3^{-1})\to\tau_2$. Define

$$\mathcal{K}_1=\mathcal{S}_\Gamma\cap\mathcal{S}_{A'}\backslash\mathcal{S}_\Gamma/\mathcal{S}_\Gamma\cap\mathcal{S}_{B'},$$

$$\mathcal{K}_2 = \mathcal{S}_\Delta \cap \mathcal{S}_{A''} \backslash \mathcal{S}_\Delta / \mathcal{S}_\Delta \cap \mathcal{S}_{A'''},\ \mathcal{K}_3 = \mathcal{S}_\Lambda \cap \mathcal{S}_{B''} \backslash \mathcal{S}_\Lambda / \mathcal{S}_\Lambda \cap \mathcal{S}_{B'''},$$
$$\mathcal{T}_1 = \mathcal{S}_{A'} \cap \mathcal{S}_\Gamma / \mathcal{S}_{A'} \cap \mathcal{S}_\Gamma \cap \mathcal{S}_{B'}^{\pi_1},$$
$$\mathcal{T}_2 = \mathcal{S}_{A''} \cap \mathcal{S}_\Delta / \mathcal{S}_{A''} \cap \mathcal{S}_\Delta \cap \mathcal{S}_{A'''}^{\pi_2},\ \mathcal{T}_3 = \mathcal{S}_{B''} \cap \mathcal{S}_\Lambda / \mathcal{S}_{B''} \cap \mathcal{S}_\Lambda \cap \mathcal{S}_{B'''}^{\pi_3}.$$

Lemma 3 implies $\mathcal{M}_i = \{\nu_i \pi_i^{-1} \,|\, \pi_i^{-1} \in \mathcal{K}_i,\ \nu_i \in \mathcal{T}_i\}$ for $1 \le i \le 3$. Thus $H_{\gamma,\delta,\lambda}^{A,B}$ equals

$$\frac{1}{c_2} \sum_{\substack{\pi_i^{-1} \in \mathcal{K}_i \\ 1 \le i \le 3}} \sum_{\substack{\nu_i \in \mathcal{T}_i \\ 1 \le i \le 3}} \sum_{\substack{\tau_1 \in \mathcal{S}_A \\ \tau_2 \in \mathcal{S}_B}} F^{A,B}(\tau_1 \cdot (\nu_1 \pi_1^{-1} \times \nu_2 \pi_2^{-1} \times id_r), \tau_2 \cdot (it_t \times \nu_3 \pi_3^{-1} \times id_s)).$$

Substitute τ_1 for $\tau_1 \cdot (\nu_1 \times \nu_2 \times id_r)$ and τ_2 for $\tau_2 \cdot (id_t \times \nu_3 \times id_s)$ and define

$$c_3 = \#(\mathcal{S}_{A'} \cap \mathcal{S}_\Gamma \cap \mathcal{S}_{B'}^{\pi_1}) \cdot \#(\mathcal{S}_{A''} \cap \mathcal{S}_\Delta \cap \mathcal{S}_{A'''}^{\pi_2}) \cdot \#(\mathcal{S}_{B''} \cap \mathcal{S}_\Lambda \cap \mathcal{S}_{B'''}^{\pi_3}).$$

Then

$$H_{\gamma,\delta,\lambda}^{A,B} = \sum_{\pi_i^{-1} \in \mathcal{K}_i, 1 \le i \le 3} \sum_{\tau_1 \in \mathcal{S}_A, \tau_2 \in \mathcal{S}_B} \frac{1}{c_3} F^{A,B}(\tau_1 \cdot (id_t \times id_r \times \pi_2), \tau_2 \cdot (\pi_1 \times id_s \times \pi_3))$$

$$= \sum_{\pi_i^{-1} \in \mathcal{K}_i, 1 \le i \le 3} \frac{1}{c_3} \sum_{\tau_1 \in \mathcal{S}_A^{id_t \times id_r \times \pi_2}} \sum_{\tau_2 \in \mathcal{S}_B^{\pi_1 \times id_s \times \pi_3}} F^{A,B}((id_t \times id_r \times \pi_2) \cdot \tau_1, (\pi_1 \times id_s \times \pi_3) \cdot \tau_2).$$

Applying Lemma 11 to an admissible quintuple $(\underline{t}, \underline{r}, \underline{s}, A^{id_t \times id_r \times \pi_2}, B^{\pi_1 \times id_r \times \pi_3})$ we obtain $\gamma_0 \vdash \underline{t}$, $\delta_0 \vdash \underline{r}$, $\lambda_0 \vdash \underline{s}$, $\rho_1 \in \mathcal{S}_\Gamma$, $\rho_2 \in \mathcal{S}_\Delta$ and $\rho_3 \in \mathcal{S}_\Lambda$ such that the octuple $(\underline{t}, \underline{r}, \underline{s}, A^{\rho_1 \times \rho_2 \times \pi_2 \rho_2}, B^{\pi_1 \rho_1 \times \rho_3 \times \pi_3 \rho_3}, \gamma_0, \delta_0, \lambda_0)$ is admissible, $\mathcal{S}_{\Gamma_0} = \mathcal{S}_{A'}^{\rho_1} \cap \mathcal{S}_\Gamma \cap \mathcal{S}_{B'}^{\pi_1 \rho_1}$, $\mathcal{S}_{\Delta_0} = \mathcal{S}_{A''}^{\rho_2} \cap \mathcal{S}_\Delta \cap \mathcal{S}_{A'''}^{\pi_2 \rho_2}$ and $\mathcal{S}_{\Lambda_0} = \mathcal{S}_{B''}^{\rho_3} \cap \mathcal{S}_\Lambda \cap \mathcal{S}_{B'''}^{\pi_3 \rho_3}$. If we set $\rho_I = \rho_1 \times \rho_2 \times \rho_2$, $\rho_{II} = \rho_1 \times \rho_3 \times \rho_3$, $\pi_I = id_t \times id_r \times \pi_2$, and $\pi_{II} = \pi_1 \times id_s \times \pi_3$, then

$$H_{\gamma,\delta,\lambda}^{A,B} = \sum_{\pi_i^{-1} \in \mathcal{K}_i, i \in 1,3} \frac{1}{c_3} \sum_{\tau_1 \in \mathcal{S}_A^{\pi_I \rho_I}, \tau_2 \in \mathcal{S}_B^{\pi_{II} \rho_{II}}} F^{A,B}(\pi_I \rho_I \tau_1, \pi_{II} \rho_{II} \tau_2).$$

By Lemma 2, there are permutations $\eta_1 \in \mathcal{S}_A^{\pi_I \rho_I}$ and $\eta_2 \in \mathcal{S}_B^{\pi_{II} \rho_{II}}$ that satisfy the conditions $A\langle \pi_I \rho_I(i) \rangle = A^{\pi_I \rho_I} \langle \eta_1(i) \rangle$, $B\langle \pi_{II} \rho_{II}(j) \rangle = B^{\pi_{II} \rho_{II}} \langle \eta_2(j) \rangle$ for $1 \le i \le t + 2r$, $1 \le j \le t + 2s$. Hence

$$F^{A,B}(\pi_I \rho_I \tau_1, \pi_{II} \rho_{II} \tau_2) = \mathrm{sgn}(\pi_1 \pi_2 \pi_3 \tau_1 \tau_2) \prod_{i=1}^{t} x^{T|i|}_{A^{\pi_I \rho_I} \langle \eta_1 \tau_1(i) \rangle, B^{\pi_{II} \rho_{II}} \langle \eta_2 \tau_2(i) \rangle}$$

$$\prod_{j=1}^{r} y^{R|j|}_{A^{\pi_I \rho_I}\langle \eta_1\tau_1(t+j)\rangle, A^{\pi_I\rho_I}\langle \eta_1\tau_1(t+r+j)\rangle} \prod_{k=1}^{s} z^{S|k|}_{B^{\pi_{II}\rho_{II}}\langle \eta_2\tau_2(t+k)\rangle, B^{\pi_{II}\rho_{II}}\langle \eta_2\tau_2(t+s+k)\rangle}.$$

Finally, substitute τ_1 for $\eta_1\tau_1$, and τ_2 for $\eta_2\tau_2$. Using Proposition 3 and the equality $c_3 = \#\mathcal{S}_{\Gamma_0}\,\#\mathcal{S}_{\Delta_0}\,\#\mathcal{S}_{\Lambda_0}$ we derive the required assertion. $\triangle$

Corollary 2 and Proposition 4 show that for any admissible quintuple $(\underline{t}, \underline{r}, \underline{s}, A_0, B_0)$ the element $H^{A,B}_{\gamma,\delta,\lambda}$ belongs to the span over $\mathbb{Z}$ of the set

$$\{\mathrm{DP}^{A,B}_{r,s} \mid \text{a quintuple } (\underline{t}, \underline{r}, \underline{s}, A, B) \text{ is admissible}\}.$$

Remark 3 *Using Lemma 6 together with reasoning from the proof of Remark 1 it is easy to see that the last statement remains valid over a field $\mathcal{K}$ of positive characteristic.*

Proposition 2 concludes the proof of Theorem 2.

7 An example

We will apply Theorem 2 to the following example.

Let $\mathcal{Q}$ be a zigzag-quiver with $l_1 = 0$, $l_2 = 1$, $m = 2$ (see Section 5). For short, we write d for d_3. Depict $\mathcal{Q}$ schematically as

$$\mathcal{K}^2 \quad \bullet \quad \xrightarrow{1_-,\ldots,d_-} \quad \bullet \quad (\mathcal{K}^2)^*$$

The space of mixed representations is $\mathcal{R} = (\mathcal{K}^{2\times 2})^d$, where, as usual, $\mathcal{K}^{2\times 2}$ is the space of all 2×2 matrices over $\mathcal{K}$. The group $\mathcal{G} = SL(2)$ acts on the coordinate ring $\mathcal{K}[\mathcal{R}] = \mathcal{K}[z^k_{ij} \mid 1 \le i, j \le 2,\ 1 \le k \le d]$ by the formula: $g \cdot Z_k = g^T Z_k g$, $g \in \mathcal{G}$. Note that $\mathcal{R}$ corresponds to d-tuple of bilinear forms on $\mathcal{K}^2$ (see Section 1 for details). Invariants of pairs of bilinear forms were investigated in [1].

Consider the action of $GL(2)$ on $\mathcal{K}[\mathcal{R}]$ by $g \cdot Z_k = g^{-1} Z_k g$, $g \in GL(2)$. Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Define an automorphism of algebras $\Phi : \mathcal{K}[\mathcal{R}] \to \mathcal{K}[\mathcal{R}]$ such that $\Phi(z^k_{ij})$ is equal to the (i, j)-th entry of $Z_k J$.

Proposition 5 *The restriction of Φ to $GL(2)$-invariants is an isomorphism of algebras*

$$\mathcal{K}[\mathcal{R}]^{GL(2)} \cong \mathcal{K}[\mathcal{R}]^{\mathcal{G}}.$$

Proof. The algebra $\mathcal{K}[\mathcal{R}]^{GL(2)}$ is known to be generated by $\det(Z_k)$, $\mathrm{tr}(Z_{k_1}\ldots Z_{k_r})$, where $1 \leq k, k_1, \ldots, k_r \leq d$ (see [9]). Denote by $\mathcal{A}(d)$ the $\mathcal{K}$-algebra generated by $\det(Z_k)$, $\mathrm{tr}(Z_{k_1}J\ldots Z_{k_r}J)$. We have to prove $\mathcal{K}[\mathcal{R}]^{\mathcal{G}} = \mathcal{A}(d)$.

Note that for $g \in \mathcal{G}$ we have $gJg^T = J$ and

$$g\cdot \mathrm{tr}(Z_{k_1}J\cdot Z_{k_2}J\cdot \ldots \cdot Z_{k_r}J)$$

$$= \mathrm{tr}(g^T Z_{k_1} gJ\cdot g^T Z_{k_2} gJ\cdot \ldots \cdot g^T Z_{k_r} gJ) = \mathrm{tr}(Z_{k_1}J\cdot Z_{k_2}J\cdot \ldots \cdot Z_{k_r}J).$$

This proves the inclusion $\mathcal{A}(d) \subset \mathcal{K}[\mathcal{R}]^{\mathcal{G}}$.

Theorem 2 implies that $\mathcal{K}[\mathcal{R}]^{\mathcal{G}}$ is generated by $\mathrm{DP}^{0,B}_{0,0,\underline{s}}$, where $B = (B_1,\ldots,B_q)$, $\underline{s} = (s_1,\ldots,s_d)$, and a quintuple $(0,0,\underline{s},0,B)$ is admissible. So $q = s = s_1+\ldots+s_d$. Our goal is to show that $\mathrm{DP}^{0,B}_{0,0,\underline{s}}$ is a polynomial with integer coefficients in the above mentioned generators of $\mathcal{A}(d)$. Obviously, without loss of generality we can assume $\mathcal{K} = \mathbb{Q}$.

Let $\lambda_{max} \vdash \underline{s}$ be the partition given in part (iii) of Definition 3, and let S, Λ_{max} be the distributions determined by $\underline{s}, \lambda_{max}$, respectively. Since $\Lambda_{max} = B'' \cap B''' \cap S$, the following formula holds:

$$\#\mathcal{S}_{\Lambda_{max}} = \prod_{k=1}^{d} \prod_{1\leq i,j\leq s} \#\{l \in [1,s] \,:\, B|l| = i,\ B|l+s| = j,\ S|l| = k\}!.$$

We denote the right hand side of the above equality by $c(B,\underline{s})$.

For an arbitrary $\underline{s} \in \mathbb{N}^d$ and a distribution $B = (B_1,\ldots,B_s)$ of $[1,2s]$ with $\#B_1 = \ldots = \#B_s = 2$ define

$$\mathrm{P}^B_{\underline{s}} = \frac{1}{c(B,\underline{s})} \sum_{\tau\in\mathcal{S}_B} \prod_{l=1}^{s} \mathrm{sgn}(\tau) z^{S|l|}_{B\langle\tau(l)\rangle, B\langle\tau(s+l)\rangle}.$$

By Proposition 3, we have $\mathrm{DP}^{0,B}_{0,0,\underline{s}} = \pm \mathrm{P}^B_{\underline{s}}$ for the admissible quintuple $(0,0,\underline{s},0,B)$.

A distribution B is called *decomposable*, if there is a proper subset $\mathcal{Q}$ of $[1,s]$ such that for $1 \leq l \leq s$ we have $B|l|, B|s+l| \in \mathcal{Q}$, or $B|l|, B|s+l| \notin \mathcal{Q}$. If B is decomposable, then there are $\underline{s}_r$, B_r and a mapping $\psi_r : [1,d] \to [1,d]$ (where r is 1 or 2) that satisfy $\mathrm{P}^B_{\underline{s}} = \overline{\psi}_1(\mathrm{P}^{B_1}_{\underline{s}_1})\overline{\psi}_2(\mathrm{P}^{B_2}_{\underline{s}_2})$. Here the homomorphism of algebras

$\overline{\psi}_r : \mathcal{K}[\mathcal{R}] \to \mathcal{K}[\mathcal{R}]$ is given by substitution $\overline{\psi}_r(z_{ij}^k) = z_{ij}^{\psi_r(k)}$. Therefore without loss of generality we can assume that B is indecomposable.

The indecomposability of B implies that $c(B, \underline{s})$ is 1 or 2. Moreover, $c(B, \underline{s}) = 2$ if and only if $s = 2$, $B = (\{1,2\},\{3,4\})$, and in that case $\mathrm{P}_{\underline{s}}^B = \det(Z_1)$. Therefore we can assume $c(B, \underline{s}) = 1$.

If $s = 1$, then $B = (\{1,2\})$ and $\mathrm{P}_{\underline{s}}^B = z_{12}^1 - z_{21}^1 = -\mathrm{tr}(Z_1 J)$. Hence we can assume $s > 1$. Let D be a distribution of $[1, 2s]$ defined by: $D_1 = \{1, 2s\}$, $D_l = \{l, s+l-1\}$, where $2 \le l \le s-1$, $D_s = \{s, 2s-1\}$. Then

$$\mathrm{P}_{\underline{s}}^D = \sum_{\tau_1,\ldots,\tau_s \in \mathcal{S}_2} \mathrm{sgn}(\tau_1 \ldots \tau_s) z_{\tau_1(1),\tau_2(2)}^{S|1|} z_{\tau_2(1),\tau_3(2)}^{S|2|} \ldots z_{\tau_s(1),\tau_1(2)}^{S|s|}.$$

For $\beta = (\beta_1, \ldots, \beta_d)$ with β_k equals 1 or 2, define homomorphism of $\mathcal{K}$-algebras $\psi_\beta : \mathcal{K}[\mathcal{R}] \to \mathcal{K}[\mathcal{R}]$ by

$$\psi_\beta(z_{ij}^k) = \left\{ \begin{array}{ll} z_{ij}^k, & \text{if } \beta_k = 1 \\ z_{ji}^k, & \text{if } \beta_k = 2. \end{array} \right.$$

It is not difficult to see that there is a β and a mapping $\psi : [1, d] \to [1, d]$ such that $\mathrm{P}_{\underline{s}}^B = \pm\overline{\psi}(\psi_\beta(\mathrm{P}_{\underline{s}}^D))$.

For $1 \le k \le d$ define $U_k = Z_k J$ and denote by u_{ij}^k the entries of U_k. Hence $u_{ij}^k = (-1)^j z_{i,\xi(j)}^k$, where $\xi = (12) \in \mathcal{S}_2$ is a transposition. We obtain

$$\mathrm{tr}(U_{S|1|} \ldots U_{S|s|}) = \sum_{1 \le i_1,\ldots,i_s \le 2} u_{i_1 i_2}^{S|1|} u_{i_2 i_3}^{S|2|} \ldots u_{i_s i_1}^{S|s|}.$$

For $1 \le l \le s$ define a permutation τ_l on $\{1, 2\}$ by $\tau_l(1) = i_l$. Thus $\tau_l(2) = \xi(i_l)$ and $\mathrm{sgn}(\tau_l) = -(-1)^{i_l}$. This implies

$$\mathrm{tr}(U_{S|1|} \ldots U_{S|s|}) = (-1)^s \mathrm{P}_{\underline{s}}^D.$$

Summarizing, for an indecomposable distribution B we have shown that $\mathrm{P}_{\underline{s}}^B = \pm\mathrm{tr}(V_{k_1} J \ldots V_{k_s} J)$, where V_k is Z_k or Z_k^T. The formulas

$$\mathrm{tr}(H_1^T J M_2) = \mathrm{tr}(H_1 J M_2) - \mathrm{tr}(H_1 J)\mathrm{tr}(H_2), \quad \mathrm{tr}(H_1^T J) = -\mathrm{tr}(H_1 J),$$

valid for 2×2 matrices H_1, H_2, conclude the proof. $\triangle$

Remark 4 *Proposition 5 can not be proven by elementary methods, because for all $g, h \in SL(2)$ the inequality $g^{-1} Z_1 J g \ne h^T Z_1 h$ holds. Moreover, if $g, h \in SL(2)$ and $\mathcal{B} \subset \mathcal{K}^{2\times 2}$ consists of elements H that satisfy $g^{-1} H J g = h^T H h$, then the dimension of the closure of $\mathcal{B}$ in Zariski topology is strictly less than 4.*

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